Phys 450

Spring 2021

HW # 3 Solutions

4.9 a) # states of 3 spin $\frac{1}{2}$ = $2^3 = 8$

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{1}{2}\otimes(1\oplus 0) = 1\otimes\frac{1}{2}\oplus\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}$$

mult $= 4+2+2 = 8$

(b) $2\otimes\frac{1}{2}\otimes\frac{1}{2} = 2\otimes\left(\frac{3}{2}\oplus\frac{1}{2}\right)$

$$= \left[\frac{7}{2}\oplus\frac{5}{2}\oplus\frac{3}{2}\oplus\frac{1}{2}\right]\oplus\left[\frac{5}{2}\oplus\frac{3}{2}\right]$$

mult $5\times 3\times 2 = 30 = 8+6+4+2+6+4$

4.11 $\Delta^{++} \to p + \pi^0$ combine $\frac{1}{2}\otimes 0 = \frac{1}{2}$

S: $\frac{3}{2}$ $\frac{1}{2}$ 0

$\frac{3}{2} = l\otimes\frac{1}{2}$ $l = 2, 1$

4.12 Clebsch Gordon coefficients give

$$|3/2, 1/2\rangle = \sqrt{\tfrac{1}{3}}\,|1, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|0, \tfrac{1}{2}\rangle$$

prob. to measure electron $S = \frac{1}{2}$ $= \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$

<u>4.17</u> $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

eigenvalue problem

$$(\hat{S}_y)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\hbar}{2}\varepsilon\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{bmatrix} -\varepsilon & -i \\ i & -\varepsilon \end{bmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \qquad \det[\,] = 0$$

$$\varepsilon^2 - 1 = 0 \; ; \; \varepsilon = \pm 1$$

Eigenvectors are $\chi^y_{\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = C_+ \chi^y_+ + C_- \chi^y_-$$

with $C_+ = (\chi^y_+)^*\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}(1, -i)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}(\alpha - i\beta)$

$C_- = (\chi^y_-)^*\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}(1, i)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}(\alpha + i\beta)$

so $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}(\alpha - i\beta)\chi^y_+ + \frac{1}{\sqrt{2}}(\alpha + i\beta)\chi^y_-$

probabilities are $P_{\pm} = \frac{1}{2}|\alpha \mp i\beta|^2$

4.19

a) $\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1}$

$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1}$

$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1}$ unit matrix

b)

$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z$

$\sigma_y\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z$

$\sigma_y\sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x$

$\sigma_z\sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma_x$

$\sigma_z\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y$

$\sigma_x\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y$

all together $\boxed{\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k}$

4.20 (a) subtract $\sigma_i\sigma_j - \sigma_j\sigma_i = i\varepsilon_{ijk}\sigma_k - i\varepsilon_{jik}\sigma_k$

$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$

$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2}$

4,20 (b)

add $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

(c) $$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \left(\sum_i \sigma_i a_i\right)\left(\sum_j \sigma_j b_j\right)$$

$$= \sum_{i,j} \sigma_i \sigma_j a_i b_j$$

$$= \sum_{i,j} \left(\delta_{ij} + i\sum_k \varepsilon_{ijk}\sigma_k\right) a_i b_j$$

$$= \sum_i a_i b_i + i \sum_{ijk} \varepsilon_{ijk} a_i b_j \sigma_k$$

recognize $C_k = \sum_{i,j} \varepsilon_{ijk} a_i b_j = (a \times b)_k$

we get $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

(4.21) $U(\theta) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}$

Taylor expand

$$U(\theta) = 1 - i\frac{\vec{\theta}\cdot\vec{\sigma}}{2} + \frac{1}{2!}\left(-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}\right)^2 + \frac{1}{3!}\left(-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}\right)^3 + \cdots$$

$$\left|\vec{\theta}\cdot\vec{\sigma}\right|^2 = \vec{\theta}\cdot\vec{\theta} = \theta^2$$

$$U(\theta) = \left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \cdots\right) - i\left(\hat{\theta}\cdot\vec{\sigma}\right)\left(\theta - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots\right)$$

$$= \cos\frac{\theta}{2} - i\left(\hat{\theta}\cdot\vec{\sigma}\right)\sin\frac{\theta}{2}$$