phys 450 Spring 2021

HW #6 Solutions

#6.1

$$dN = -\Gamma N\, dt$$

integrate $\quad N(t) = N(0)\, e^{-\Gamma t}$

mean decay time $\tau = \left\langle \frac{N(t)}{N_0} \right\rangle_t = \dfrac{\int_0^\infty t\, e^{-\Gamma t}\, dt}{\int_0^\infty e^{-\Gamma t}\, dt}$

$$\tau = \frac{1}{\Gamma} \frac{\int_0^\infty x e^{-x} dx}{\int_0^\infty e^{-x} dx} = \frac{1}{\Gamma}$$

note: integration by parts gives $\int_0^\infty x^n e^{-x} dx = n!$

#6.6 Estimate with $m = \alpha\, m_\pi$ statistical factor $= \frac{1}{2}$

$$\Gamma = \frac{1}{2}\left(\frac{1}{16\pi m_\pi}\right) m_\pi^2 \alpha^2 = \frac{1}{32\pi} m_\pi \alpha^2$$

$$c\tau = \frac{\hbar c\,(32\pi)}{m_\pi c^2 \alpha^2} = \frac{(137)^2 (200\ \text{eV nm})\, 32\pi}{140 \times 10^6\ \text{eV}} = 2.69\ \text{nm}$$

$$\tau = 9 \times 10^{-18}$$

Exp value is 8.4×10^{-17} estimate off by factor 10

#6.7 $1+2\to 3+4$ scattering in CM frame

flux factor $\sqrt{(\tilde{P}_1\cdot\tilde{P}_2)^2 - m_1^2 m_2^2}$

$\tilde{P}_1 = (E_1, \vec{P}_1)$ $\tilde{P}_2 = (E_2, \vec{P}_2)$ with $\vec{P}_2 = -\vec{P}_1$

$\tilde{P}_1\cdot\tilde{P}_2 = E_1E_2 + P_1^2$

$$(\tilde{P}_1\cdot\tilde{P}_2)^2 = E_1^2E_2^2 + 2P_1^2E_1E_2 + P_1^4$$
$$= (m_1^2+p^2)(m_2^2+p^2) + 2P_1^2E_1E_2 + P_1^4$$
$$= m_1^2m_2^2 + P_1^2(m_1^2+m_2^2) + 2P_1^4 + 2P_1E_1E_2$$

$$(\tilde{P}_1\cdot\tilde{P}_2)^2 - m_1^2m_2^2 = P_1^2\left[m_1^2+m_2^2+2P_1^2+2E_1E_2\right]$$
$$= P_1^2\left[E_1+E_2+2E_1E_2\right] = P_1^2(E_1+E_2)^2$$

In lab frame,

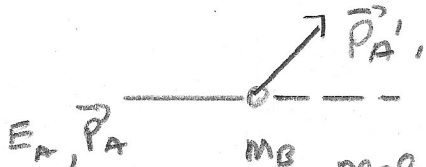
$\tilde{P}_1 = (E_1, \vec{P})$ $\tilde{P}_2 = (m_2, \vec{0})$

$\tilde{P}_1\cdot\tilde{P}_2 = E_1 m_2$

$$(\tilde{P}_1\cdot\tilde{P}_2)^2 - m_1^2m_2^2 = E_1^2m_2^2 - m_1^2m_2^2 = (p^2+m_1^2)m_2^2 - m_1^2m_2^2$$
$$= p^2m_2^2$$

$\boxed{\#6.8}$

$A + B \to A + B$



neglect recoil, $|\vec{p}_B{}'| \approx 0$

flux factor $= p_A m_B$

$$d\sigma = \frac{|m|^2}{4}\left(\frac{1}{p_A m_B}\right)\frac{(2\pi)^4}{(2\pi)^6}\,\frac{d^3\vec{p}_A{}'\, d^3p_B'}{2E_A'\, E_B'}\,\delta^4(p_A + p_B - p_A' - p_B')$$

using $\delta^3(\vec{p}_A + \vec{p}_B{}' - \vec{p}_A{}' - \vec{p}_B)$ for d^3p_B' integration

$$d\sigma = \frac{|m|^2}{64\pi^2}\left(\frac{1}{p_A m_B}\right)\left(\frac{d\Omega\, p_A'^2\, dp_A'}{E_A' m_B}\right)\delta^0(E_A - E_A')$$

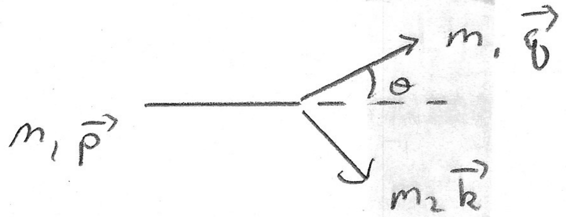
$$p_A'^2 = E_A'^2 + m_A^2 \Rightarrow p_A' dp_A' = E_A' dE_A'$$

$$\int \frac{p_A' dp_A'}{E_A' m_A}\,\delta^0(\,) = \int \frac{p_A' dE_A'}{m_B}\,\delta^0(E_A - E_A') = \frac{p_A}{m_B}$$

so $\displaystyle\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_B^2}\right)|m|^2$

[6.10] elastic scattering in lab

(a)



$$d\sigma = |\mathcal{M}|^2 \frac{1}{4\mathcal{F}} (2\pi)^4 \delta^4(\ ) \frac{d^3 q}{(2\pi)^3 2E_q} \frac{d^3 k}{(2\pi)^3 2E_k}$$

$$\mathcal{F}^2 = (E_p m_2)^2 - (m_1 m_2)^2 = (p^2 + m_1^2)(m_2^2) - (m_1 m_2)^2 = p^2 m_2^2$$

$$\delta^3(\vec{p} - \vec{q} - \vec{k}) \Rightarrow \vec{k} = \vec{p} - \vec{q}$$

$$E_k = \sqrt{m_2^2 + |\vec{p} - \vec{q}|^2}$$

$$d\sigma = \frac{|\mathcal{M}|^2}{4^2 p m_2} \frac{1}{(2\pi)^2} \frac{d^3 q}{4 E_q E_k} \delta^0(E_p + m_2 - E_q - E_k)$$

$$E_p + m_2 = E_q + E_k = \sqrt{q^2 + m_1^2} + \sqrt{|\vec{p} - \vec{q}|^2 + m_2^2}$$

$$\frac{dE_p}{dq} = \frac{q}{E_q} + \frac{q - p\cos\theta}{E_k} = \frac{q E_k + q E_q - E_q p \cos\theta}{E_q E_p}$$

$$= \frac{1}{E_q E_p}\left[q(E_p + m_2) - E_q p \cos\theta\right]$$

$$\frac{d^3 q}{E_q E_k} = \frac{d\Omega_q\, q^2\, dq}{E_q E_k} = \frac{d\Omega_q\, q^2\, dE_p}{q(E_p + m_2) - E_q p \cos\theta}$$

$$\boxed{\frac{d\sigma}{d\Omega_q} = |\mathcal{M}|^2 \frac{1}{64\pi^2} \frac{q^2}{p\, m_2} \left[\frac{1}{q(E_p + m_2) - E_q p \cos\theta}\right]}$$

(b) with $m_1 = 0$ $\quad p = E_p$, $q = E_q$

$$\frac{d\sigma}{d\Omega_q} = |M|^2 \frac{1}{64\pi^2} \frac{q}{p m_2} \left[\frac{1}{p + m_2 - q\cos\theta} \right]$$

use $\quad p + m_2 = E_k + q$

and $\quad E_k^2 = m_2^2 + |\vec{p} - \vec{q}|^2$

$$(p - q + m_2)^2 = m_2^2 + p^2 + q^2 - 2pq\cos\theta$$

$$(p-q)^2 + 2m_2(p-q) + m_2^2 = m_2^2 + p^2 + q^2 - 2pq\cos\theta$$

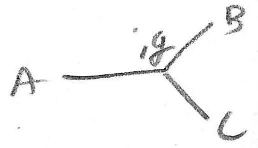
$$-pq + m_2(p-q) = -pq\cos\theta$$

$$\frac{m_2(p-q)}{q} = p - \cos\theta$$

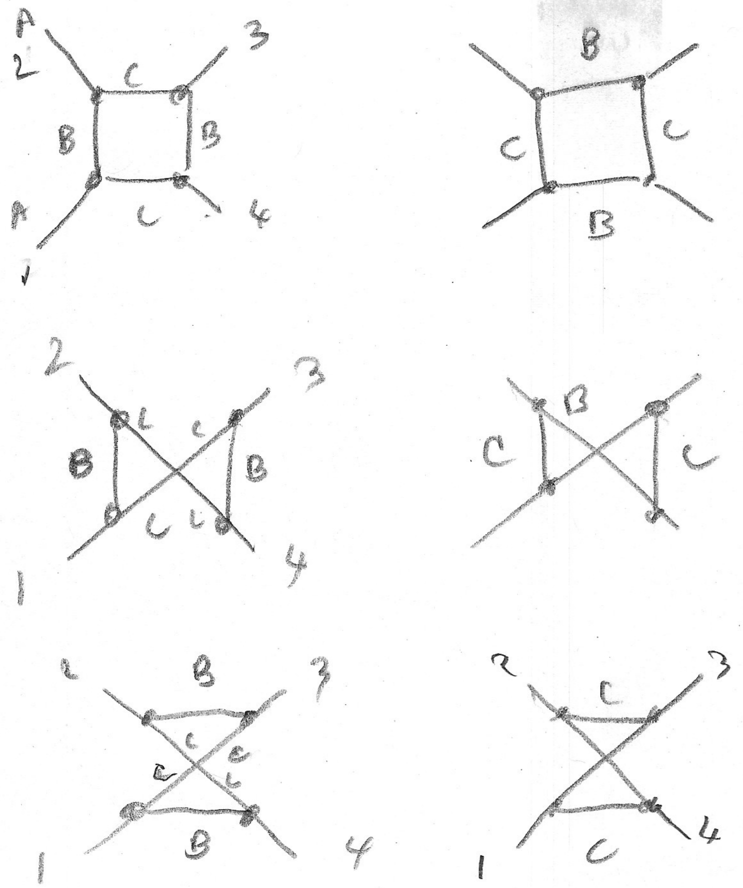
and so $\quad \frac{m_2 p}{q} = p + m_2 - \cos\theta$

$$\frac{d\sigma}{d\Omega_q} = |M|^2 \frac{1}{64\pi^2} \left(\frac{q}{m_2 p} \right)^2$$

$\boxed{6.12}$ (a) $A + A \to A + A$ where interaction is

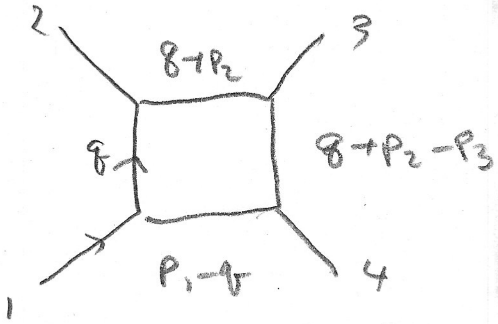


4 vertices to first order

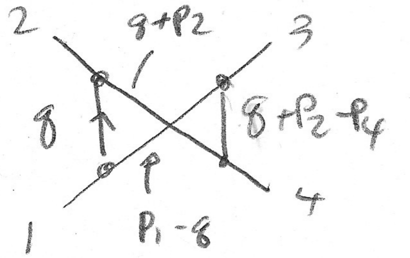


with $m_g = m_c$, each row of diagram is the same

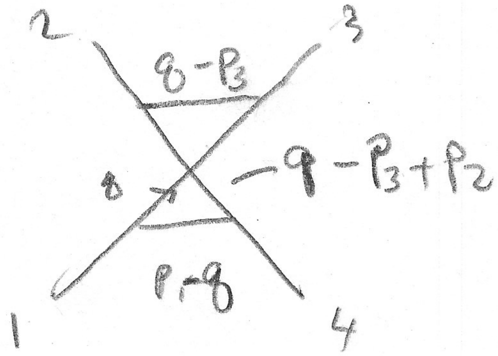
(b)



$$-iM_1 = 2(ig)^4 \int d^4q \frac{i}{q^2} \frac{i}{(p_1-q)^2} \frac{i}{(q+p_2)^2} \frac{i}{(q+p_2-p_3)^2}$$



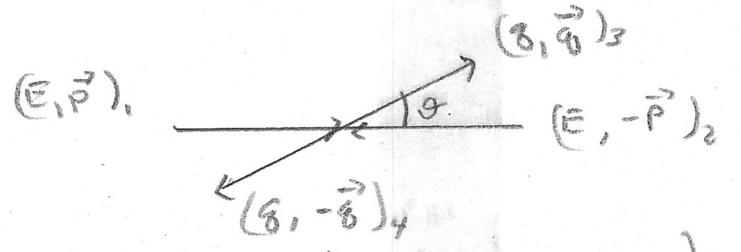
$$-iM_2 = 2(ig)^4 \int d^4q \frac{i}{q^2} \frac{i}{(p_1-q)^2} \frac{i}{(q+p_2)^2} \frac{i}{(q+p_2-p_4)^2}$$



$$-iM_3 = 2(ig)^4 \int d^4q \frac{i}{q^2} \frac{i}{(p_1-q)^2} \frac{i}{(q-p_3)^2} \frac{i}{(q-p_3+p_2)^2}$$

total amplitude is $M = M_1 + M_2 + M_3$

6.13 A+A → B+B in CM frame with $m_B = m_C = 0$



eq. 6.50 $\mathcal{M} = g^2 \left[\frac{1}{(P_4 - P_2)^2} + \frac{1}{(P_3 - P_2)^2} \right]$

$$|P_4 - P_2|^2 = (q - E, -\vec{q} - \vec{p})^2 = (q-E)^2 - |\vec{q} + \vec{p}|^2$$
$$= -2qE + E^2 - p^2 - 2pq\cos\theta = m_A^2 - 2qE - 2pq\cos\theta$$

$$|P_3 - P_2|^2 = m_A^2 - 2qE + pq\cos\theta$$

eq. 6.42

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{4E^2} \left(\frac{q}{p}\right) g^4 \left(\frac{1}{m_A^2 - 2qE}\right)^2 \left[\frac{1}{1-ac} + \frac{1}{1+ac}\right]^2$$

$$a = \frac{2pq}{m_A^2 - 2qE}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{4E^2}\right)\left(\frac{q}{p}\right) g^4 \frac{a^2}{4p^2q^2} \left(\frac{2}{1-a^2c^2}\right)^2$$

$$= \frac{g^4}{256\pi^2} \frac{1}{E^2} \frac{1}{qp^3} \frac{a^2}{(1-a^2c^2)^2} \xrightarrow[\substack{m_A=0 \\ a=-1 \\ q=p=E}]{} \frac{g^4}{256\pi^2} \frac{1}{p^6} \left(\frac{1}{\sin^4\theta}\right)$$

$$\sigma = \frac{g^4}{256\pi^2} \frac{1}{E^2} \frac{a^2}{qp^3} (2\pi) \int_{-1}^{+1} \frac{dc}{(1-a^2c^2)^2}$$

$$= \frac{g^4}{128\pi} \frac{1}{E^2 q p^3} a^2 \left[\frac{\tanh^{-1}(a)}{2a} + \frac{1}{1-a^2}\right]$$

σ diverges in limit $m_A = 0$, $a = -1$