Physics 450 Spring 2021

HW #7 - Solutions

#7.1)

$$X^{\mu'} = \Lambda^{\mu}{}_{\nu} X^{\nu}$$

in 1 time, 1 space

$$\Lambda^{\mu}{}_{\nu} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

$$\eta^{\mu\nu} = \mathrm{diag}(1,-1) = \eta_{\mu\nu}$$

$$X'_{\mu} = \eta_{\mu\nu} X'^{\nu}$$ lower index

lower index transforms as →

$$X'_{\mu} = \eta_{\mu\alpha} \Lambda^{\alpha}{}_{\nu} X^{\nu} = \boxed{\eta_{\mu\alpha} \Lambda^{\alpha}{}_{\beta} \eta^{\beta\nu}} X_{\nu}$$

$$\bar{\eta} \cdot \bar{\Lambda} \cdot \bar{\eta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -\beta \\ 1 & -\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ -\beta & -1 \end{pmatrix}$$

$$= \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} = (\bar{\Lambda})^{-1}$$

$$X^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} X'^{\nu}$$

$$\frac{\partial X^{\mu}}{\partial X'^{\nu}} = (\Lambda^{-1})^{\mu}{}_{\nu}$$

so $$\frac{\partial \phi}{\partial X'^{\mu}} = \frac{\partial \phi}{\partial X^{\mu}} \left(\frac{\partial X^{\mu}}{\partial X'^{\nu}} \right) = \frac{\partial \phi}{\partial X^{\mu}} (\Lambda^{-1})^{\mu}{}_{\nu}$$

transforms as lower index 4-vector:

$$\partial_{\mu} \phi \equiv \frac{\partial \phi}{\partial X^{\mu}}$$

7.4

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ (p_x - i p_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

$$u^{(1)\dagger} u^{(2)} = \frac{N^2}{(E+m)^2} \Big(p_z(p_x - i p_y) - p_z(p_x - i p_y) \Big) = 0$$

$$u^{(3)} = N \begin{pmatrix} p_z/(E-m) \\ (p_x + i p_y)/E-m \\ 1 \\ 0 \end{pmatrix} \quad u^{(4)} = N \begin{pmatrix} (p_x - i p_y)/(E-m) \\ -p_z/(E-m) \\ 0 \\ 1 \end{pmatrix}$$

$$u^{(3)\dagger} u^{(4)} = \frac{N^2}{E-m} \left[p_z(p_x - i p_y) - p_z(p_x - i p_y) \right] = 0$$

$$u^{(1)\dagger} u^{(3)} = N^2 \left[\frac{p_z}{(E-m)} + \frac{p_z}{E+m} \right] \neq 0$$

$\boxed{7.5}$ lower components of $u^{(1)}, u^{(2)}$ with $\vec{p} = m\gamma v \hat{z}$ in N.R. limit

$$\frac{p_z}{E+m} = \frac{\gamma v}{\gamma + 1} \xrightarrow[\substack{\text{N.R. limit} \\ \gamma = 1}]{} \frac{v}{2}$$

So $u^{(1)} \rightarrow \begin{pmatrix} \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \\ \frac{v}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$ $\quad u^{(2)} \rightarrow \begin{pmatrix} \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \\ -\frac{v}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$

7.8 (a) $\gamma \cdot p - m = 0$

$(\gamma^0)^2 = \bar{1}$ so $(\gamma^0)^2 p^0 - \gamma_0 \vec{\gamma} \cdot \vec{p} - \gamma_0 m = 0$

$p^0 = \gamma_0 \vec{\gamma} \cdot \vec{p} + \gamma_0 m = \hat{H}$ $\quad p^0 = i \frac{\partial}{\partial t}$; $\vec{p} = \frac{1}{i} \vec{\nabla}$

(b) $\hat{L}^i = \varepsilon^i{}_{jk} x^j p^k$ repeated indicai summed

$[\hat{H}, \hat{L}^i] = [\gamma^0 \gamma^\ell p^\ell , \varepsilon^i{}_{jk} x^j p^k]$

$= \gamma^0 \varepsilon^i{}_{jk} \gamma^\ell p^k [p^\ell, x^j]$
$\qquad \delta^{\ell j} \left(\frac{1}{i}\right)$

$= \frac{1}{i} \gamma^0 \varepsilon^i{}_{jk} \gamma^j p_k$

$[\hat{H}, \hat{\vec{L}}] = \frac{1}{i} \gamma^0 \vec{\gamma} \times \vec{p}$

(c) $\hat{\vec{S}} = \frac{1}{2} \vec{\Sigma}$ $\quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma}, & 0 \\ 0, & \vec{\sigma} \end{pmatrix}$

$\vec{\gamma} = \begin{pmatrix} 0, & \vec{\sigma} \\ -\vec{\sigma}, & 0 \end{pmatrix}$ $\quad \gamma^0 \vec{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0, & \vec{\sigma} \\ -\vec{\sigma}, & 0 \end{pmatrix} = \begin{pmatrix} 0, & \vec{\sigma} \\ \vec{\sigma}, & 0 \end{pmatrix}$

$[\gamma^0 \gamma^i, \Sigma^j] = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} - \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \sigma_i \sigma_j \\ \sigma_i \sigma_j, & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \sigma_i \\ \sigma_j \sigma_i, & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0 \end{pmatrix}$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

$$[\gamma^0\gamma^i, \Sigma^j] = 2i\,\varepsilon_{ijk}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = 2i\,\varepsilon_{ijk}\,\gamma^0\gamma^k$$

$$[\hat{H}, \hat{S}^j] = \frac{p^i}{2}[\gamma^0\gamma^i, \Sigma^j]$$

$$= i\,p^i\,\varepsilon_{ijk}\,\gamma^0\gamma^k = i(\gamma^0\vec{\gamma}\times\vec{p})^j$$

$$[\hat{H}, \hat{\vec{S}}] = i\gamma^0\vec{\gamma}\times\vec{p}$$

$$[\hat{H}, \vec{L}+\vec{S}] = -i\gamma^0\vec{\gamma}\times\vec{p} + i\gamma^0\vec{\gamma}\times\vec{p} = 0$$

(d) $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$

$$\hat{S}^2 = \frac{1}{4}\Sigma^2 = \frac{1}{4}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\cdot\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \frac{3}{4}\begin{pmatrix} I_{2\times2} & 0 \\ 0 & I_{2\times2} \end{pmatrix}$$

$$\vec{\sigma}\cdot\vec{\sigma} = \sigma_i\sigma_i = \delta_{ii}I_{2\times2} = 3\,I_{2\times2}$$

$I_{2\times2}$ is 2×2 unit matrix

$$\hat{S}^2\psi = \frac{3}{4}\psi = \frac{1}{2}\left(\frac{1}{2}+1\right)\psi$$

7.9

$\psi_c = i\gamma^2 \psi^*$

$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$U^{(1)} = N \left(1, 0, \frac{P_z}{E+m}, \frac{P_x + iP_y}{E+m}\right)^T$

$U_c^{(1)} = i\gamma_2 \left(1, 0, \frac{P_z}{E+m}, \frac{P_x - iP_y}{E+m}\right)^T$

$= \left(\frac{P_x - iP_y}{E+m}, \frac{-P_z}{E+m}, 0, 1\right)^T = V^{(1)}$

$U_2^{(2)} = \left(0, 1, \frac{P_x - iP_y}{E+m}, -\frac{P_z}{E+m}\right)^T$

$U_2^{(c)} = i\gamma_2 \left(0, 1, \frac{P_x + iP_y}{E+m}, \frac{-P_z}{E+m}\right)^T$

$= \left(\frac{-P_z}{E+m}, \frac{-P_x - iP_y}{E+m}, -1, 0\right)^T = V^{(2)}$

7.11)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \; ; \; (\Lambda^{-1})^\mu{}_\nu x'^\nu = x^\mu$$

$$\frac{\partial x^\mu}{\partial x'^\nu} = (\Lambda^{-1})^\mu{}_\nu$$

$$\psi'(x') = \hat{S}_x \psi(x)$$

$$\hat{S}_x (i\gamma^\mu \partial_\mu \psi - m\psi) = 0$$

$$i \hat{S}_x \gamma^\mu \partial_\mu \psi - m\psi' = 0$$

so $\hat{S}_x \gamma^\mu \partial_\mu \psi = \gamma^\mu \partial'_\mu \hat{S}_x \psi = \gamma^\mu \partial'_\mu \psi'$

$$\hat{S}_x \gamma^\mu \partial_\mu = \gamma^\mu \partial'_\mu \hat{S}_x$$

multiply by x^ν

$$\hat{S}_x \gamma^\nu = \gamma^\mu \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) \hat{S}_x = \gamma^\mu (\Lambda^{-1})^\nu{}_\mu \hat{S}_x$$

$$\gamma^\nu = \hat{S}_x^{-1} (\gamma^\mu \Lambda^{-1}{}^\nu{}_\mu) S_x$$

Boost in $\hat{x}$ direction $v = \tanh\theta$

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}' = \underbrace{\begin{pmatrix} \cosh\theta, & \sinh\theta \\ \sinh\theta, & \cosh\theta \end{pmatrix}}_{(\Lambda^{-1})^\nu{}_\mu} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

7, 11 continued

then $\gamma^\mu \Lambda^{-1\,\nu}{}_\mu$ $\qquad c \equiv \cosh\theta$, $\; s \equiv \sinh\theta$

$$(\gamma^0, \gamma^1)\begin{pmatrix} c & s \\ s & c \end{pmatrix} = \begin{pmatrix} c\gamma^0 + s\gamma^1 \\ s\gamma^0 + c\gamma^1 \end{pmatrix}$$

$$\begin{pmatrix} \gamma^0 \\ \gamma^1 \end{pmatrix} = \hat{S}_x^{-1} \begin{pmatrix} c\gamma^0 + s\gamma^1 \\ s\gamma^0 + c\gamma^1 \end{pmatrix} \hat{S}_x$$

$$\gamma^1 \left(\hat{S}_x \gamma^0 = (c\gamma^0 + s\gamma^1)\hat{S}_x \right)$$

$$\gamma^0 \left(\hat{S}_x \gamma^1 = (s\gamma^0 + c\gamma^1)\hat{S}_x \right)$$

$$\gamma^1 \hat{S}_x \gamma^0 = (c\gamma^1\gamma^0 - s)\hat{S}_x$$

$$\gamma^0 \hat{S}_x \gamma^1 = (s + c\gamma^0\gamma^1)\hat{S}_x$$

add and $\gamma^1\gamma^0 + \gamma^0\gamma^1 = 0$

$$\gamma^1 \left(\gamma^1 \hat{S}_x \gamma^0 + \gamma^0 \hat{S}_x \gamma^1 = 0 \right)$$

$$\left(-\hat{S}_x \gamma^0 + \gamma^1\gamma^0 \hat{S}_x \gamma^1 = 0 \right)\gamma^1$$

$$-\hat{S}_x \gamma^0\gamma^1 + \gamma^1\gamma^0 \hat{S}_x \underbrace{(\gamma^1)^2}_{-1} = 0$$

$$\left[\hat{S}_x, \gamma^0\gamma^1\right] = 0 \quad \Rightarrow \quad \hat{S}_x = a + b\gamma^0\gamma^1$$

7.11 continued

to determine a, b

$$\hat{S}_x \gamma^0 = (c\gamma^0 + s\gamma^1)\hat{S}_x$$

$$(a + b\gamma^0\gamma^1)\gamma^0 = (c\gamma^0 + s\gamma^1)(a + b\gamma^0\gamma^1)$$

$$a\gamma^0 - b\gamma^1 = ca\gamma^0 + cb\gamma^1 + as\gamma^1 + bs\underbrace{\gamma^1\gamma^0\gamma^1}_{+\gamma^0}$$

$$= (ca + bs)\gamma^0 + (cb + as)\gamma^1$$

$$a = ca + bs \qquad \frac{a}{b}(1-c) = s$$

$$-b = cb + as \qquad \frac{a}{b}s = -1 - c$$

$$\left(\frac{a}{b}\right)^2 = \left(\frac{c+1}{c-1}\right) = \left(\frac{\cosh \theta/2}{\sinh \theta/2}\right)^2$$

$$\frac{a}{b} = \pm \frac{\cosh \theta/2}{\sinh \theta/2} \qquad \frac{b}{a} = \frac{-s}{1+c} \Rightarrow b = -\sinh \theta/2$$

$$\boxed{\hat{S}_x = \cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}\gamma^0\gamma^1}$$

7.13 (a) $\hat{S}_x = a_+ + a_- \gamma^0\gamma^1 \qquad a_\pm = \pm\sqrt{\tfrac{1}{2}(\gamma \pm 1)}$

$$(\gamma^0\gamma^1)^\dagger = (\gamma^1)^\dagger(\gamma^0)^\dagger = (-\gamma^1)(\gamma^0) = \gamma^0\gamma^1$$

$$\hat{S}_x^\dagger = \hat{S}_x$$

$$(\hat{S}_x)^2 = a_+^2 + 2a_+a_-\gamma^0\gamma^1 + a_-^2\gamma^0\gamma^1\gamma^0\gamma^1$$

$$\gamma^0\gamma^1\gamma^0\gamma^1 = -\underbrace{\gamma^0\gamma^0}_{+1}\underbrace{\gamma^1\gamma^1}_{-1} = +1$$

$$(\hat{S}_x)^2 = \underbrace{a_+^2 + a_-^2}_{\gamma} + \underbrace{2a_+a_-}_{}\gamma^0\gamma^1$$

$$-2\sqrt{\tfrac{1}{2}(\gamma+1)}\sqrt{\tfrac{1}{2}(\gamma-1)} = -\sqrt{\gamma^2-1}$$

$$= -\gamma\sqrt{1-(1-v^2)} = -\gamma v$$

$$\hat{S}_x^2 = \gamma - \gamma v \underbrace{\gamma^0\gamma^1}_{\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}}$$

(b) $\hat{S}^\dagger \gamma^0 \hat{S} = \hat{S}\gamma^0(a_+ + a_-\gamma^0\gamma^1)$

$$= (a_+ + a_-\gamma^0\gamma^1)(a_+\gamma^0 + a_-\gamma^1)$$

$$a_+^2\gamma^0 + a_+a_-(\gamma^1 + \underbrace{\gamma^0\gamma^1\gamma^0}_{-\gamma^1}) + (a_-)^2\gamma^0\underbrace{\gamma^1\gamma^1}_{-1}$$

$$= \gamma^0(a_+^2 - a_-^2) = \gamma^0$$

$\underline{7.14|}$ with $\psi = S_x^{-1}\psi'$

$S_x^{-1} = \cosh\theta/2 - \sinh\frac{\theta}{2}\gamma^0\gamma^1$

$\bar{\psi}\gamma 5\psi = \psi^+\gamma^0\gamma 5\psi$

Hermitian

$\psi^+ = (S_x^{-1}\psi')^+ = (\psi')^+(\hat{S}_x^{-1})^+ = (\psi')^+ S_x^{-1}$

$\bar{\psi}\gamma 5\psi = \psi'^+ \hat{S}_x^{-1}\gamma^0\gamma 5\hat{S}_x^{-1}\psi'$

$S_x^{-1}\gamma^0\gamma^5 = \gamma^0\gamma^5\cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}\gamma^0\gamma^1\gamma^0\gamma^5$

$\gamma^0\gamma^1\gamma^0\gamma^5 = \gamma^0\gamma^5\gamma^1\gamma^0 = -\gamma^0\gamma^5\gamma^0\gamma^1$

$S_x^{-1}\gamma^0\gamma 5 = \gamma^0\gamma 5(\cos\theta/2 + \sin\theta/2\,\gamma^0\gamma^1)$

$= \gamma^0\gamma 5\hat{S}_x$

So $\bar{\psi}\gamma 5\psi = (\psi')^+\gamma^0\gamma 5(\hat{S}_x\hat{S}_x^{-1})\psi'$

$= \bar{\psi}'\gamma 5\psi'$

7,16] normalization $u^{+}u = 2E \Rightarrow N = \sqrt{E+m}$

$$\bar{u}^{(1)} u^{(1)} = u^{+} \gamma^0 u = (E+m) \left(1, 0, \frac{p_z}{E+m}, \frac{p_x - i p_y}{E+m}\right) \begin{pmatrix} 1 \\ 0 \\ -p_z/(E+m) \\ -\frac{p_x - i p_y}{E+m} \end{pmatrix}$$

$$= E+m \quad \frac{-p^2}{E+m} = \frac{1}{E+m}\left[E^2 + m^2 + 2mE - m^2 - E^2\right]$$

$$= \frac{2m(E+m)}{E+m} = 2m$$