Phys 450 Spring 2021

HW #8 (Ch 7) Solutions

7.18

$\hat{P}\psi = \gamma_0 \psi$ parity $\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$\hat{P}\, U_i = +U_i$ $i=1,2$

$\hat{P}\, V_i = -V_i$

Fermion, anti Fermion have opposite intrinsic parity

7.23

a) $\Lambda = i a k e^{-i p \cdot x}$ $(\hbar = 1)$

$\Box \Lambda = p \cdot p \, \Lambda$ $\Box = \partial^\mu \partial_\mu$ 4-divergence

$p \cdot p = E_0^2 - |\vec{p}|^2 = 0$ for photons

(b) $A_\mu' = A_\mu + \partial_\mu \Lambda = A_\mu + a k e^{-i p \cdot x}$

$A_\mu = a e^{-i p \cdot x} \varepsilon^\mu(p)$

So $A_\mu' = a e^{-i p \cdot x} (\varepsilon^\mu + k p^\mu) = a e^{-i p \cdot x} (\varepsilon'^\mu)$

for $k = \frac{\varepsilon^0}{p^0}$ $\varepsilon'^0 = 0$.

(7.24)

$$U^1 = N\begin{pmatrix} \chi_+ \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_+ \end{pmatrix} \qquad \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$$

$$U^2 = N\begin{pmatrix} \chi_- \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_- \end{pmatrix} \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

$$N = \sqrt{E+m}$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \bar{U}^{(1)} = N\begin{pmatrix} \chi_+^+ & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_+^+ \end{pmatrix}$$

$$\chi_+\chi_+^+ = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \equiv I^U$$

$$I^U + I^D = I_{2\times 2}$$

$$\chi_-\chi_-^+ = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \equiv I^D$$

$$U^1\bar{U}^1 = E+m\begin{pmatrix} I^U & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}I^U \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}I^U & -\underbrace{\frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2}}_{=\frac{E-m}{E+m}}I^U \end{pmatrix}$$

$$= \begin{pmatrix} (E+m)I^U & -\vec{\sigma}\cdot\vec{p}\, I^U \\ \vec{\sigma}\cdot\vec{p}\, I^U & (-E+m)I^U \end{pmatrix}$$

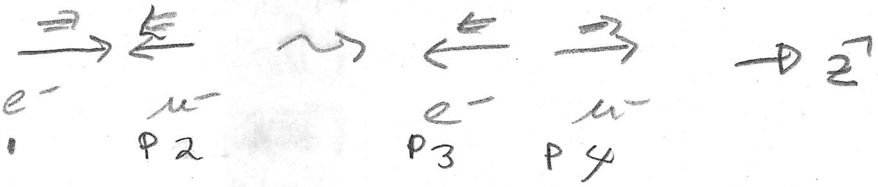
$$U^2\bar{U}^2 = \begin{matrix} (E+m)I^D, & -\vec{\sigma}\cdot\vec{p}\, I^D \\ \vec{\sigma}\cdot\vec{p}\, I^D, & (-E+m)I^D \end{matrix}$$

Adding

$$U^1\bar{U}^1 + U^2\bar{U}^2 = \begin{pmatrix} (E+m)I & -\vec{\sigma}\cdot\vec{p}\, I \\ (\vec{\sigma}\cdot\vec{p})I & (-E+m)I \end{pmatrix} = E\gamma^0 - \vec{p}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} + mI_{4\times 4}$$

$$= \gamma^\mu p_\mu + mI_{4\times 4}$$

$\boxed{\# 7.26}$



$$u_1 = \begin{pmatrix} \chi^+ \\ p\chi^+ \end{pmatrix} \qquad u_3 = \begin{pmatrix} \chi^- \\ +p\chi^- \end{pmatrix}$$

$$u_2 = \begin{pmatrix} \chi^- \\ -p\chi^- \end{pmatrix} \qquad u_4 = \begin{pmatrix} \chi^+ \\ -p\chi^+ \end{pmatrix}$$

$$\mathcal{M} = \frac{-g^2}{(p_1-p_3)^2} (\bar{u}_3 \gamma^\mu u_1)(\bar{u}_4 \gamma_\mu u_2)$$

$p_1 = (E, p\hat{z}) \qquad (p_1-p_3)^2 = (0, 2p\hat{z})^2 = -4p^2$

$p_3 = (E, -p\hat{z})$

$\bar{u}_3 \gamma^\mu u_1 = u_3^\dagger \gamma^0 \gamma^\mu u_1 = u_3^\dagger (1, \gamma^0 \vec{\gamma}) u_1$

$u_3^\dagger u_1 = 0$ since $\chi^- \chi^+ = 0$

$\gamma^0 \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

$\bar{u}_3 \gamma^\mu u_1 = \left((\chi^-)^\dagger, +p(\chi^-)^\dagger \right) \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi^+ \\ p\chi^+ \end{pmatrix} = 2p(\hat{x} + i\hat{y})$

$\vec{\sigma}\chi^+ = \hat{x}\chi^- - i\hat{y}\chi^- + \hat{z}\chi^+$

and $\vec{\sigma}\chi^- = \hat{x}\chi^+ + i\hat{y}\chi^+ + \hat{z}\chi^-$

so $\bar{u}_4 \gamma^\mu u_2 = -2p(\hat{x} - i\hat{y})$

giving $\mathcal{M} = \dfrac{-g^2}{-4p^2} (2p)(\hat{x}+i\hat{y})(-2p)(\hat{x}-i\hat{y}) = -2g^2$

7.28 *Casimir for anti-particle*

$$G = (\bar{V}_a \Gamma_1 V_b)(\bar{V}_a \Gamma_2 V_b)^*$$

$$(\bar{V}_a \Gamma_2 V_b)^* = (V_a^+ \gamma^0 \Gamma_2 V_b)^*$$

$$= V_b^+ \Gamma_2^+ \gamma_0 V_a = \bar{V}_b \bar{\Gamma}_2 V_a$$

$$\bar{\Gamma}_2 = \gamma^0 \Gamma_2^+ \gamma^0$$ as before for particle

$$\sum_s V_s(p) \bar{V}_s(p) = \not{p} - m$$

$$\sum_{b\ \text{spin}} G = \bar{V}_a \Gamma_1 \sum_{b\ \text{spin}} V_b \bar{V}_b \bar{\Gamma}_2 V_a$$

$$= \bar{V}_a \Gamma_1 (\not{p}_b - m_b) \bar{\Gamma}_2 V_a \equiv \bar{V}_a Q V_a$$

$$= \sum_{i,j} (\bar{V}_a)_i Q_{ij} (V_a)_j$$

$$\sum_{a,b\ \text{spin}} G = \sum_{ij} Q_{ij} \sum_{a\ \text{spin}} (V_a)_j (\bar{V}_a)_i$$

$$= \sum_{ij} Q_{ij} (\not{p}_a - m_a)_{ji}$$

$$= \sum_i [Q(\not{p}_a - m_a)]_{ii} = \mathrm{tr}(Q(\not{p}_a - m_a))$$

$$= \mathrm{tr}\{\Gamma_1 (\not{p}_b - m_b) \bar{\Gamma}_2 (\not{p}_a - m_a)\}$$

We can see:

$$\sum_{spin} (\bar{u}(a)\, \Gamma_1 v_b)(\bar{u}_a \Gamma_2 v_b)$$

$$= \mathrm{tr}\left\{ \Gamma_1 (\not{p}_a + m_a)\, \bar{\Gamma}_2 (\not{p}_b - m_b) \right.$$

and

$$\sum_{spin} (\bar{v}_a \Gamma_1 v_b)(\bar{v}_a \Gamma_2 v_b)^*$$

$$= \mathrm{tr}\left\{ \Gamma_1 (\not{p}_a - m_a)\, \overline{\Gamma}_2 (\not{p}_b + m_b) \right.$$

7.29 (a) $\gamma^0 \gamma^{\nu +} \gamma^0 = \gamma^0 \quad \nu = 0$

$\nu = 1,2,3$: $\gamma^0 (\gamma^i)^+ \gamma^0 = -\gamma^0 \gamma^i \gamma^0 = \gamma^i$

So $\gamma^0 \gamma^{\nu +} \gamma^0 = \gamma^\nu$

(b) $\bar{\Gamma} = \gamma_0 \Gamma^+ \gamma_0$

$= \gamma_0 (\gamma_a \gamma_b \cdots \gamma_c)^+ \gamma_0$

$= \gamma_0 (\gamma_c \cdots \gamma_b \gamma_a) \gamma_0$

$= \underbrace{\gamma_0 \gamma_c \gamma_0} \cdots \underbrace{\gamma_0 \gamma_b \gamma_0} \gamma_0 \gamma_a \gamma_0$

$= \gamma_c \cdots \gamma_b \gamma_a$ reverse order

$\boxed{7.31}$

$$\mathrm{tr}(A+B) = \sum_i (A_{ii} + B_{ii}) = \mathrm{tr}\,A + \mathrm{tr}\,B \checkmark$$

$$\mathrm{tr}(\alpha A) = \sum_i \alpha A_{ii} = \alpha\, \mathrm{tr}(A) \checkmark$$

$$\mathrm{tr}(AB) = \sum_{ij} A_{ij} B_{ji} = \sum B_{ji} A_{ij} = \mathrm{tr}(BA) \checkmark$$

$$g_{\mu\nu} g^{\mu\nu} = g_{\mu\nu} g^{\nu\mu} = \mathrm{tr}\left[\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right]$$

$$= \mathrm{tr}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = 4 \checkmark$$

$$\not{a}\not{b} + \not{b}\not{a} = a_\mu b_\nu \gamma^\mu \gamma^\nu + b_\nu a_\mu \gamma^\nu \gamma^\mu$$

$$= a_\mu b_\nu [\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] = 2 a_\mu b_\nu g^{\mu\nu} = 2(a \cdot b)$$

$\boxed{7.32}$

$$(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu})\, g_{\mu\nu}$$

$$\gamma^\mu \gamma_\mu + \gamma_\mu \gamma^\mu = 2 \cdot 4$$

$$\gamma_\mu \gamma^\mu = 4 \checkmark$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu})\gamma^\mu$$

$$= -4\gamma^\nu + 2\gamma^\nu = -2\gamma^\nu \checkmark$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = (-\gamma^\nu\gamma_\mu + 2g_\mu{}^\nu)(-\gamma^\mu\gamma^\lambda + 2g^{\lambda\mu})$$

$$= 4\gamma^\nu\gamma^\lambda - 2\gamma^\nu\gamma^\lambda - 2\gamma^\nu\gamma^\lambda + 4g_\mu{}^\nu g^{\lambda\mu}$$

$$= 4g^{\lambda\nu} = 4g^{\nu\lambda} \checkmark$$

7.32 continued

$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu =$

$- \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu \gamma^\sigma + \gamma_\mu \gamma^\nu \gamma^\lambda (2g^{\mu\sigma})$

$= -4 g^{\nu\lambda} \gamma^\sigma + 2 \gamma_\sigma \gamma^\nu \gamma^\lambda$

$= -4 g^{\nu\lambda} \gamma^\sigma - 2 \gamma_\sigma \gamma^\lambda \gamma^\nu + 2 \gamma_\sigma (2 g^{\nu\lambda})$

$= -2 \gamma_\sigma \gamma^\lambda \gamma^\nu$ ✓

7', $\gamma_\mu (a_\nu \gamma^\nu) \gamma^\mu = a_\nu (-2\gamma_\nu) = -2 \not{a}$

8' $\gamma_\mu (\not{a} \not{b}) \gamma^\mu = a_\rho b_\sigma \gamma_\mu \gamma^\rho \gamma^\sigma \gamma^\mu$

$= a_\rho b_\sigma (4 g^{\rho\sigma}) = 4\, a \cdot b$

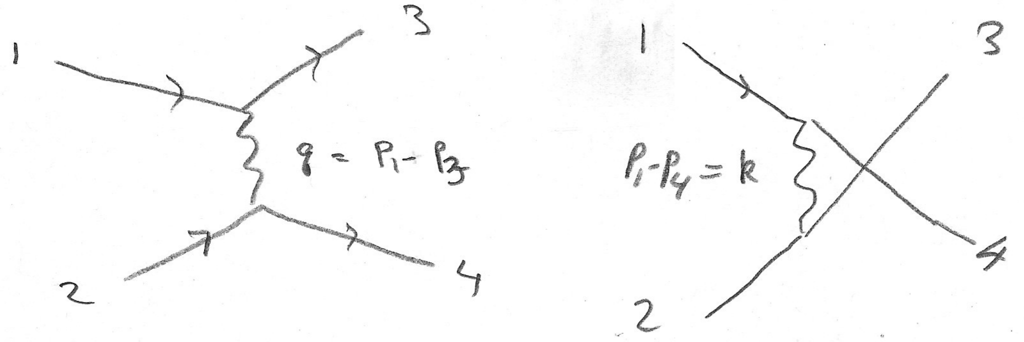
9' $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = a_\nu b_\lambda c_\sigma \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu$

$= - a_\nu b_\lambda c_\sigma (-2 \gamma^\sigma \gamma^\lambda \gamma^\nu)$

$= -2 \not{c} \not{b} \not{a}$

7.37 $\quad e^- e^- \to e^- e^-$

high energy , $m_e \approx 0$



m_1 $\qquad$ m_2

multiplied by -1

$$-i M_1 = \bar{U}_3 (-ie\gamma^\mu) U_1 \frac{-ig^{\mu\nu}}{q^2} \bar{U}_4 (-ie\gamma^\nu) U_2$$

$$M_1 = \frac{+e^2}{q^2} (\bar{U}_3 \gamma^\mu U_1)(\bar{U}_4 \gamma_\mu U_2)$$

and

$$M_2 = (-1)\left(\frac{+e^2}{k^2}\right)(\bar{U}_4 \gamma^\mu U_1)(\bar{U}_3 \gamma_\mu U_2)$$

7.37 continued

$$\langle m_1 m_2^* \rangle = -\frac{1}{4} \sum \frac{e^4}{g^2 k^2} (\bar{u}_3 \gamma^\mu u_1)(\bar{u}_4 \gamma_\mu u_2)$$

$$(\bar{u}_4 \gamma^\nu u_1 \bar{u}_3 \gamma_\nu u_2)^*$$

$$(\ )^* = \bar{u}_2 \gamma_\nu u_3 \bar{u}_1 \gamma^\nu u_4$$

$$\langle m_1 m_2^* \rangle = -\frac{1}{4} \frac{e^4}{g^2 k^2} \sum (\bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu \underbrace{u_2 \bar{u}_2}_{\not{p}_2} \gamma_\nu u_3 \bar{u}_1 \gamma^\nu u_4)$$

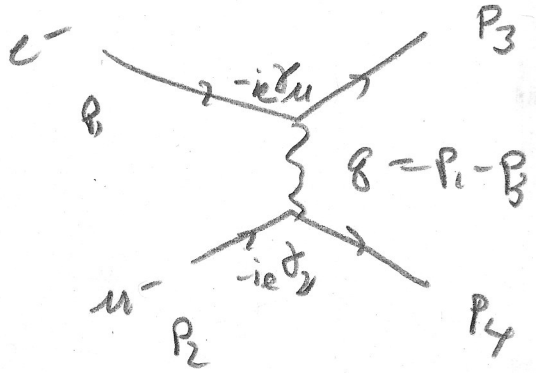
$$= -\frac{1}{4} \frac{e^4}{g^2 k^2} \sum \bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu \not{p}_2 \gamma_\nu u_3 \bar{u}_1 \gamma^\nu u_4$$

$$= -\frac{1}{4} \frac{e^4}{g^2 k^2} \sum \underbrace{u_3 \bar{u}_3}_{\not{p}_3} \gamma^\mu \underbrace{u_1 \bar{u}_1}_{\not{p}_1} \gamma^\nu \underbrace{u_4 \bar{u}_4}_{\not{p}_4} \gamma_\mu \not{p}_2 \gamma_\nu$$

$$= -\frac{1}{4} \frac{e^4}{k^2 g^2} \mathrm{tr} (\not{p}_3 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu)$$

$\boxed{7.38}$ $e^- \mu^- \to e^- \mu^-$ high energy limit ($m_e \approx 0$, $m_\mu \approx 0$)

(a)



$$\mathcal{M} = \frac{-e^2}{q^2}(\bar{u}_3 \gamma^\mu u_1)(\bar{u}_4 \gamma_\mu u_2)$$

initial spin average, final spin sum

$$\langle \mathcal{M}^2 \rangle = \frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^4}{4q^4}|\bar{u}_3 \gamma^\mu u_1|^2 |\bar{u}_4 \gamma_\mu u_2|^2$$

$$= \frac{e^4}{4q^4}(\bar{u}_1 \gamma^\mu u_3 \bar{u}_3 \gamma^\nu u_1)(\bar{u}_2 \gamma_\mu u_4 \bar{u}_4 \gamma_\nu u_2)$$

$$= \frac{e^4}{4q^4} Tr(\gamma^\mu \not{p}_3 \gamma^\nu \not{p}_1) Tr(\gamma_\mu \not{p}_4 \gamma_\nu \not{p}_2)$$

$$Tr(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\rho) = 4(g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\lambda\nu} - g^{\mu\nu} g^{\lambda\rho})$$

$$(Tr)_{31} = 4(P_3^\mu P_1^\nu + P_1^\mu P_3^\nu - g^{\mu\nu}(P_3 \cdot P_1)$$

$$(Tr)_{42} = 4(P_{4\mu} P_{2\nu} + P_{2\mu} P_{4\nu} - g_{\mu\nu}(P_4 \cdot P_2)$$

$\boxed{7.38}$ [a] continued

$$\langle m^2\rangle = \frac{4e^4}{g^4}\Big(P_3\cdot P_4\, P_1\cdot P_2 + \underline{P_3\cdot P_2\, P_1\cdot P_4} - P_3\cdot P_2\, P_4\cdot P_2$$

$$+ \underline{P_1\cdot P_4\, P_3\cdot P_2} + P_1\cdot P_2\, P_3\cdot P_4 - P_1\cdot P_3\, P_4\cdot P_2$$

$$- P_4\cdot P_2\, P_3\cdot P_1 - P_2\cdot P_4\, P_3\cdot P_1 + 4\, P_3\cdot P_1\, P_4\cdot P_2\Big)$$

$$= \frac{8e^4}{g^4}\Big(P_1\cdot P_2\, P_3\cdot P_4 + P_1\cdot P_4\, P_3\cdot P_2\Big)$$

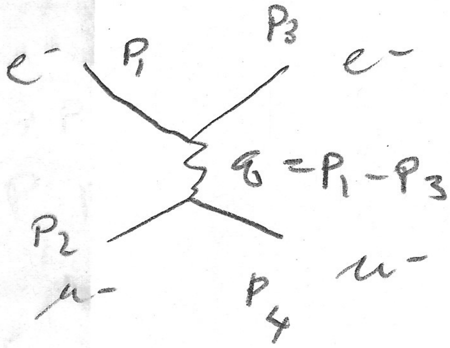
7.38 (b)

e-μ scattering

high energy limit

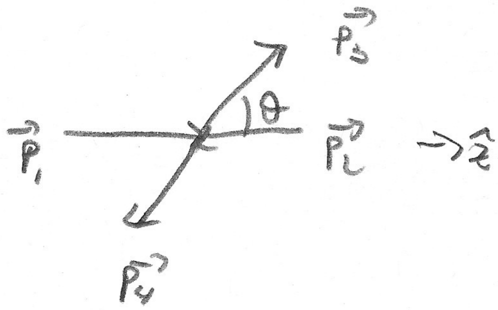
$m_e \approx 0$

$m_\mu \approx 0$



$$\langle M^2 \rangle = \frac{8e^4}{q^4}\left(P_1 \cdot P_2 \; P_3 \cdot P_4 + P_1 \cdot P_4 \; P_3 \cdot P_2\right)$$

cm frame



$P_1 = (p, p\hat{z})$

$P_2 = (p, -p\hat{z})$

$P_3 = (p', \vec{p}')$

$P_4 = (p', -\vec{p}')$

$P_1 \cdot P_2 = 2p^2$

$P_3 \cdot P_4 = 2p^2$

$P_1 \cdot P_4 = p^2 + \vec{p}' \cdot \vec{p} = p^2(1+\cos\theta)$

$P_3 \cdot P_2 = p^2(1+\cos\theta)$

$$q^2 = (P_1 - P_3)^2 = -|\vec{p} - \vec{p}'|^2 = -2p^2(1-\cos\theta)$$

$$\langle M^2 \rangle = \frac{8e^4}{4p^4(1-c)^2}\left(4p^4 + p^4(1+c)^2\right) = 2e^4\left[\frac{1+\cos^4\theta/2}{\sin^4\theta/2}\right]$$

7.38 continued

$$d\sigma = \langle M^2 \rangle \frac{1}{4(P_1 \cdot P_2)} (2\pi)^4 \delta^4() \frac{d^3 P_3}{(2\pi)^3 P_3} \frac{d^3 P_4}{(2\pi)^3 2 P_4}$$

$$\delta^3(\vec{P_1} + \vec{P_2} - \vec{P_3} - \vec{P_4}) \Rightarrow \vec{P_3} = -\vec{P_4}$$

$$d\sigma = \langle m^2 \rangle \frac{1}{4(P_1 \cdot P_2)} \left(\frac{1}{2\pi}\right)^2 \frac{1}{4} \delta^0() \frac{d^3 P_3}{p'^2}$$

$$\delta^0(2p - 2p') = \frac{1}{2}\delta^0(p - p') \qquad d^3 P_3 = d\Omega\, p'^2 dp'$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \underbrace{\left(\frac{1}{P_1 \cdot P_2}\right)}_{2p^2} \frac{1}{4\pi^2} \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) \langle m^2 \rangle$$

$$= \frac{1}{4^3} \frac{1}{4p^2} \frac{1}{\pi^2} 2e^4 [\ ]$$

$$= \left(\frac{1}{8\pi}\right)^2 \frac{e^4}{2p^2} \left[\frac{1 + \cos^4 \theta/2}{\sin^4 \theta/2} \right]$$