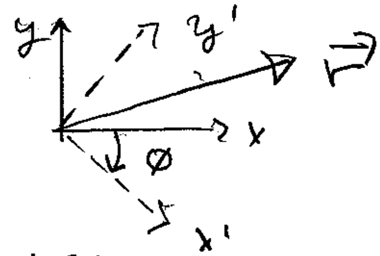
Lecture #1 Relativistic Kinematics

I. Coordinate transformations

1. Rotations



$$\vec{r} = x\hat{x} + y\hat{y} = x'\hat{x}' + y'\hat{y}'$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\phi\hat{z}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The same vector $\vec{r}$ is represented by column vector

$$\vec{r} \longrightarrow (x, y)^{T} \quad \text{or} \quad \vec{r} \longrightarrow (x', y')^{T}$$

— Transpose

"is represented by"

Note that we often use sloppy notation and write $\vec{r} = (x, y)$; $\vec{r}' = (x', y')$ to represent the same vector in different coordinates.

Rotations leave the Euclidean vector product invariant:

$$x^2 + y^2 = x'^2 + y'^2 = \vec{r} \cdot \vec{r}$$

Rotations form a group where

$$R(\phi_1\hat{z})\, R(\phi_2\hat{z}) = R((\phi_1 + \phi_2)\hat{z}) = R(\phi_2\hat{z})\, R(\phi_1\hat{z})$$

but in general $R(\vec{a})\, R(\vec{b}) \neq R(\vec{b})\, R(\vec{a})$

Still, $R(\vec{a})\, R(\vec{b}) = R(\vec{c})$ form group

2. <u>importance of invariance</u>:
Consider arbitrary linear transformation leaving Euclidean dot-product invariant.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

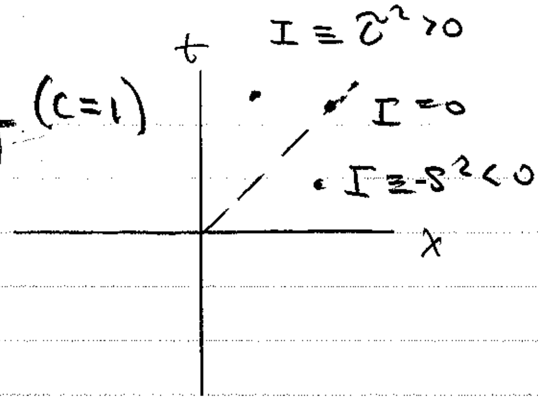
$$x'^2 + y'^2 = \left(R_{11}x + R_{12}y\right)^2 + \left(R_{21}x + R_{22}y\right)^2$$

$$= \underbrace{\left(R_{11}^2 + R_{21}^2\right)}_{=1} x^2 + \underbrace{\left(R_{12}^2 + R_{22}^2\right)}_{=1} y^2 + \underbrace{\left(R_{12}R_{11} + R_{21}R_{12}\right)}_{=0} xy$$

3 eq., 4 unknowns

easy to see that the solution is parameterized by angle ϕ as above.

3. <u>Special Relativity</u> $(c=1)$



invariant interval!

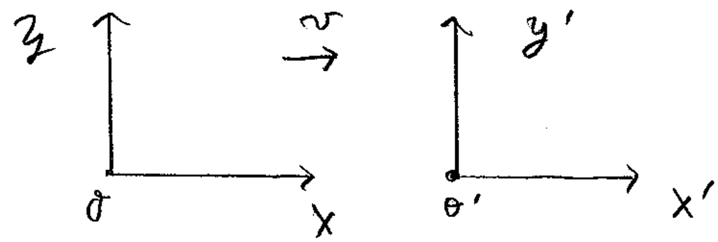
$$(\Delta t)^2 - \Delta\vec{x} \cdot \Delta\vec{x} = I$$

τ^2 = proper time
$s^2 \equiv$ proper length

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2 = (\Delta t)^2\left(1 - v^2\right) = \left(\Delta t/\gamma\right)^2$$

length contraction is a bit more subtle.
<u>Show that</u> $\Delta X = \Delta s/\gamma$

4. <u>Lorentz boost</u> (velocity transformation)
leave $t^2 - x^2$ invariant.



Origin $O' \rightarrow$ (t, vt) unprimed frame
$(0,0)$ primed frame

4-vector $\bar{x} \equiv (t, \vec{x})$

$$\bar{x}\cdot\bar{x} \equiv \left(t^2 - \vec{x}\cdot\vec{x}\right)$$

parameterized by boost "angle" θ for hyperbolic "rotation"

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$x' = t\left[-\sinh\theta + v\cosh\theta\right] = 0$$

$$\boxed{v = \tanh\theta}$$

$$\sinh\theta = v\gamma$$
$$\cosh\theta = \gamma$$

4-velocity $\quad \Delta\bar{x} = (\Delta t, \Delta\vec{x})$

divide by L.I. $\Delta\tau$: $\dfrac{\Delta\bar{x}}{\Delta\tau} = \gamma(1, \vec{v})$

nice thing about boost parameter is that two boosts in the same direction "add" like rotations

$$B_x(\theta_1)\,B_x(\theta_2) = B_x(\theta_1+\theta_2)$$

boosts in 1 direction form group

addition of velocities: $\theta' = \theta + \theta_{rel}$

$$\tanh\theta' = \frac{\tanh\theta + \tanh\theta_{rel}}{1+\tanh\theta\tanh\theta_{rel}}$$

$$v' = \frac{v+v_{rel}}{1+v\,v_{rel}}$$

Not only do boosts in different direction not commute, but they do not form a group!

in general, $B(\vec{\theta}_1)\,B(\vec{\theta}_2)$ not always another boost!

General L.T. is 4x4 matrix including <u>boost</u> and <u>rotation</u>.

physical implications such as Thomas Precession and most importantly <u>intrinsic spin</u>.

II. Kinematics

$E, \vec{p}$ are useful concepts because they are conserved.

1. momentum 4-vector $\bar{p} = (p^0, \vec{p}) = (E, \vec{p})$

$$= m\gamma(1, \vec{v})$$

m is Lorentz invariant quantity called mass or rest energy.

$$\bar{p}\cdot\bar{p} = m^2 = E^2 - \vec{p}\cdot\vec{p}$$

we have $E = m\gamma$, $\vec{p} = m\gamma\vec{v}$, $E_k = E - m = (\gamma - 1)m$

and $\gamma = E/mc^2$

2. Zero momentum frame

Like "Center of mass" frame in N.R. mechanics, processes often are simpler to understand in frame where total $\vec{p}_{TOT} = 0$. 3 momentum

Consider proton beam on nuclear target.

$\overset{\circ}{E_p} \quad \overrightarrow{\vec{p}} \qquad\qquad \dot{m}_N \quad \rightarrow x$

where $E_p = \sqrt{p^2+m^2}$ $\longrightarrow$ boost v_zm

$\bar{P}_{TOT} = (E_p + m_N, \vec{p})$ Lab

$\bar{p}^{zm}_{TOT} = (E^{zm}, \vec{0})$ ZM

$E^{zm} = \gamma^{zm}(E_p + m_N - v^{zm} p)$

$0 = \gamma^{zm}(p - v^{zm}(E_p + m_N))$

giving $v^{zm} = \dfrac{p}{E_p + m_N} = \dfrac{p}{\sqrt{p^2+m_p^2} + m_N}$

3. Ex. $p + p \to p + p + p + \bar{p}$ ($p\bar{p}$ production)

minimum (threshold) energy for process

$p^{zm}_{TOT} = (4m, \vec{0})$

$\bar{p}^{zm}_{TOT} \cdot \bar{p}^{zm}_{TOT} = 16m^2 = (E+m, \vec{p})^2 = (E+m)^2 - p^2$

$= (E+m)^2 - (E^2 - m^2)$

$\Rightarrow E = 7m$