Phys 450 Spring '07

2-1

Lecture #2

I. Cross sections & Decay Rates

primary observables in nuclear & particle physics

① decay rate $\Gamma =$ constant prob. / time

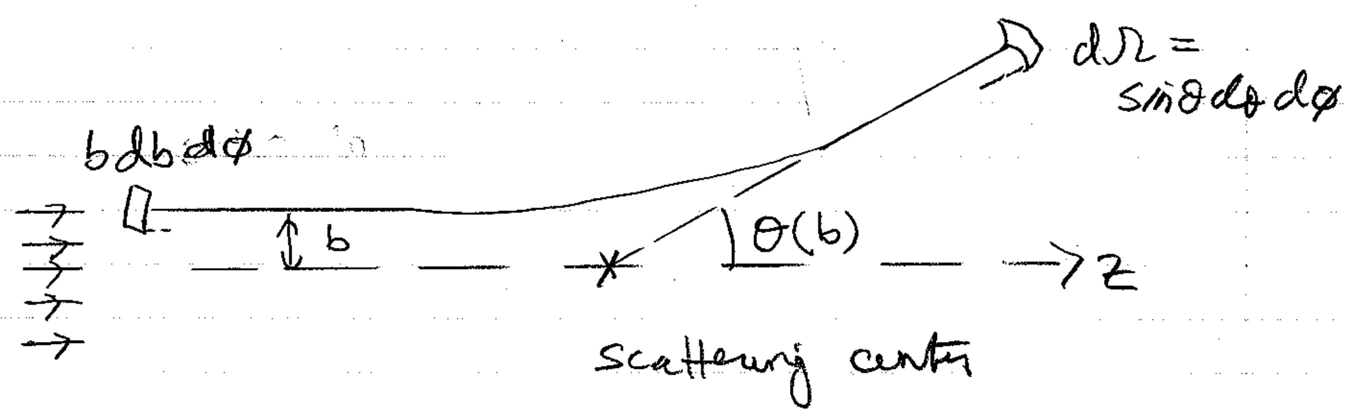
$$dN = -\Gamma N\, dt$$

$$\langle t \rangle = 1/\Gamma \equiv \tau \quad \text{"lifetime"}$$

② cross section flux normalized scattering rate

Classical: particle follow trajectory.

Scattering angle θ depends on impact parameter b



flux $F \equiv$ # incident particles / area / time

$\phi \equiv$ azimuthal angle

all particles thru $b\,db\,d\phi \rightarrow d\Omega$

$$\Delta\sigma = \frac{(\text{rate @ } \theta,\phi)\cdot\Delta\Omega}{F} = \frac{F\, b\,\Delta b\,\Delta\phi}{F}$$

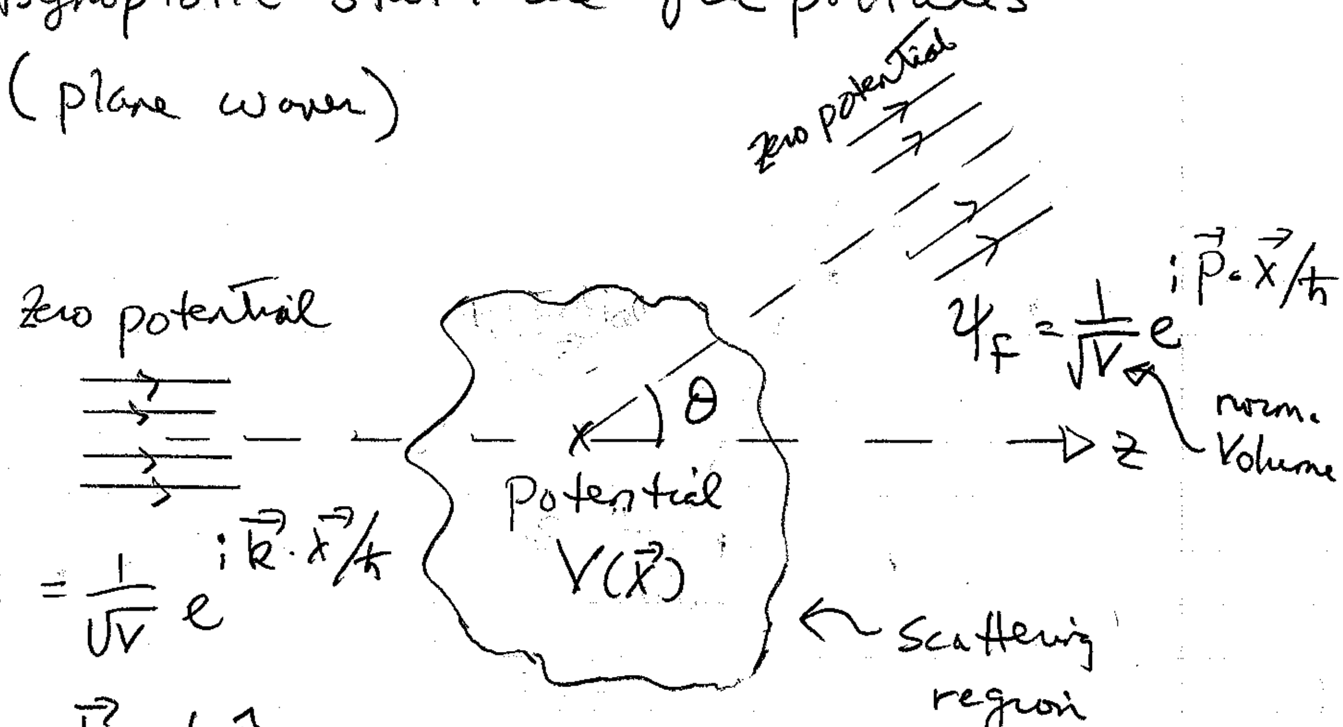
$$\frac{\Delta\sigma}{\Delta\Omega} = \frac{b\,\Delta b\,\Delta\phi}{\sin\theta\,\Delta\theta\,\Delta\phi} = \frac{b}{\sin\theta}\,\frac{\Delta b}{\Delta\theta}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

③ <u>Non-Relativistic Quantum scattering</u>

No trajectories; amplitude to scatter

Asymptotic states are free particles (plane waves)



$$\psi_i = \frac{1}{\sqrt{V}}\, e^{i\vec{k}\cdot\vec{x}/\hbar}$$

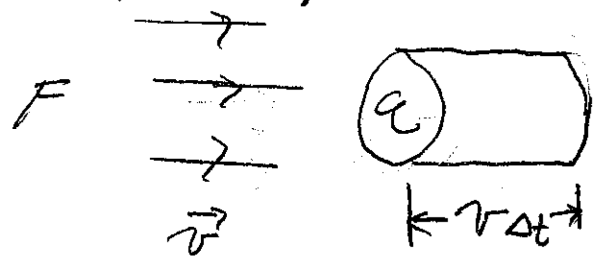
$$\vec{k} = k\hat{z}$$

$$\psi_f = \frac{1}{\sqrt{V}}\, e^{i\vec{p}\cdot\vec{x}/\hbar}$$

Amplitude M_{fi}

$$d\sigma = \frac{dR}{F}$$

Q.M. flux includes incident wave function normalization of 1 particle/volume.



incident particle speed v

$$F = \left(\frac{1}{V}\right)\frac{v\,\Delta t\, a}{a\,\Delta t} = \frac{v}{V}$$

(4) Fermi's Golden Rule transition rate dR,

$$dR = \frac{2\pi}{\hbar}\left|M_{fi}\right|^2 \underbrace{\rho(E)}$$

amplitude, dynamics — density of states, kinematics

$$\rho(E) = \frac{dN_s}{dE} = \frac{V}{(2\pi\hbar)^3}\frac{p^2\,dp\,d\Omega}{dE} = \frac{V}{(2\pi\hbar)^3}\frac{p^2}{dE/dp}\,d\Omega$$

$$\frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p}{m} = v$$

note: relativistic case is the same,

$$\frac{dE}{dp} = \frac{d}{dp}\left(\sqrt{p^2+m^2}\right) = \frac{p}{E} = \frac{m\gamma v}{m\gamma} = v$$

⑤ <u>Scattering Amplitude</u>

In first order perturbation theory:

$$\mathcal{M}^{(1)}_{fi} = \int \psi_f^* V(\vec{x})\, \psi_i \, d^3x$$

$$= \frac{1}{V} \int e^{i \vec{q}\cdot\vec{x}/\hbar}\, V(\vec{x})\, d^3x$$

where $\vec{q} \equiv \vec{k} - \vec{p}$ is momentum transfer.

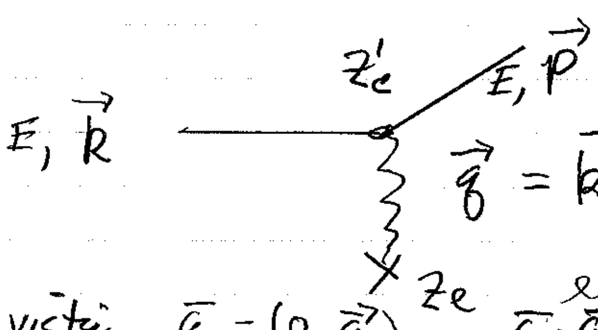
$\mathcal{M}$ is just Fourier Transform of potential

For $V(\vec{x}) = \dfrac{ZZ'e^2}{|\vec{x}|}$,

$$\mathcal{M}^{(1)}_{fi} = \frac{ZZ'e^2}{V} \int e^{i\frac{\vec{q}\cdot\vec{x}}{\hbar}} \frac{d^3x}{|\vec{x}|} = \frac{ZZ'e^2}{V} \cdot \frac{4\pi\hbar^2}{|\vec{q}|^2}$$

choose spherical coordinates with $\vec{q}$ as z-axis for integration.

Diagram representation (Feynman-like)



$\vec{q} = \vec{k} - \vec{p} \quad \propto \quad \dfrac{1}{|\vec{q}|^2}$

Note: relativistic $\bar{q} = (0, \vec{q})$ exchange of virtual photon $\bar{q}\cdot\bar{q} \equiv t \neq 0$ "off mass shell"

⑥ Rutherford Cross Section

$$d\sigma = \left(\frac{V}{v}\right)\frac{2\pi}{\hbar}\left(\frac{ZZ'e^2}{V}\right)^2\frac{(4\pi\hbar^2)^2}{q^4}\frac{V}{(2\pi\hbar)^3}\frac{p^2}{v}\,d\Omega$$

$$= 4\,\frac{p^2}{v^2}\,\frac{1}{q^4}\,(ZZ'e^2)^2$$

non-relativistic $\quad p/v = m \quad ; \; E = \frac{p^2}{2m}$

$$\vec{q}^{\,2} = |\vec{k} - \vec{p}|^2 = 2p^2(1-\cos\theta) = 4p^2\sin^2\theta/2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{m}{p^2}\right)^2\frac{(ZZ'e^2)^2}{\sin^4(\theta/2)}$$

$$= \frac{1}{(4E)^2}\frac{(ZZ'e^2)^2}{\sin^4(\theta/2)}$$

no $\hbar$ $\Rightarrow$ same as classical result.

⑦ Relativistic Cross Section

Relativistic scattering cross section is similar, written in terms of invariant amplitude $\mathcal{M}_{fi}$:
$1+2 \rightarrow 3 \cdots n+2$ (n final particles)

$$d\sigma = \underbrace{\frac{1}{2E_1 2E_2}\left(\frac{1}{v_1+v_2}\right)}_{\text{invariant flux}}|\mathcal{M}_{fi}|^2\,\underbrace{d\Phi_n}_{\text{invariant phase space}}$$

note: $d\sigma$ being area $\perp$ beam is Lorentz Invariant.

where invariant flux:

$$\bar{k}_1 = E_1 (1, \vec{v}_1)$$

$$\bar{k}_2 = E_2 (1, \vec{v}_2)$$

for $\vec{v}_1 \cdot \vec{v}_2 = -v_1 v_2$, $|\vec{v}_1 + \vec{v}_2| = [V_1^2 + V_2^2 + 2V_1 V_2]^{1/2}$

$$= V_1 + V_2$$

$$2E_1 E_2 (V_1 + V_2) = 4 \sqrt{(\bar{k}_1 \cdot \bar{k}_2)^2 - m_1^2 m_2^2}$$

invariant phase space:

$$d\Phi_n = (2\pi)^4 \delta^4 \left(\bar{k}_1 + \bar{k}_2 - \sum_{i=1}^{n} \bar{p}_i\right) \prod_{i=1}^{N} \frac{d^3 p_i}{2E_i (2\pi)^3}$$

Similarly, decay of particle w/ mass M $\mapsto$ n particles in rest frame

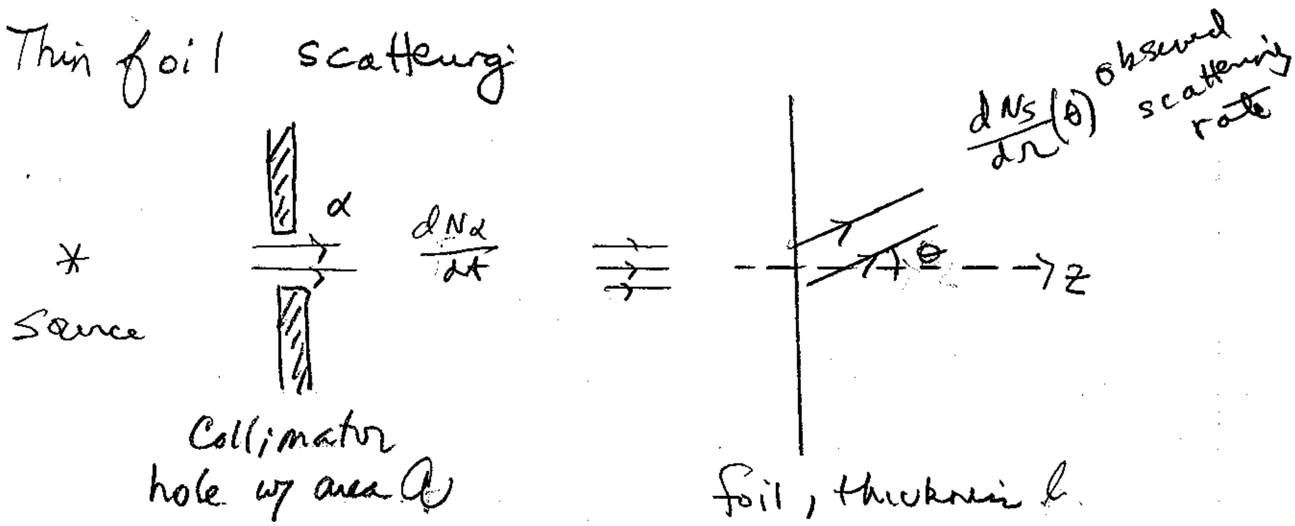
$$d\Gamma = \frac{1}{2M} |\mathcal{M}|^2 d\Phi_n$$

Formulae are nice looking after lots of massaging and moving around of factors of E, 2π.

note: $\frac{d^3 p}{E}$ is invariant.

The Rutherford Experiment

Thin foil scattering:



"thin" means ignore multiple scattering

$$F = \underbrace{\frac{dN_\alpha}{dt}}_{(\#\ \alpha\text{'s}/\text{time})\text{ thru hole}} \left(\frac{1}{a}\right) \times (\#\ \text{targets})$$

$$\#\ \text{targets} = \underbrace{\frac{A_0 \rho}{m_A}}_{\#/\text{Vol}} (l a)$$

A_0 = Avagadro's number

$m_A = A \times 10^{-3}\ \text{kg/mole} = A \times 1\ \text{gm/mole}$ molar mass

ρ = density gm/Vol

$$F = \frac{dN_\alpha}{dt} \left(\frac{A_0 \rho}{m_A}\right) l$$

Measure cross section from observed scattering rate dN_s/dt:

$$\frac{d\sigma_{exp}}{d\Omega}(\theta) = \frac{1}{F}\frac{dN_s}{dt}(\theta) = \left(\frac{m_A}{A_0 \rho \ell}\right)\left(\frac{dN_s(\theta)/dt}{dN_\alpha/dt}\right)$$

Agreement with theoretical Rutherford formula verifies theoretical assumptions:

① mass assumed infinite

$$m_N \approx M_{Atom}$$

② size is "pointlike" << 0.1 nm atomic scale

"pointlike" means follows $\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\theta/2}\Big|_{\theta=\pi} = 1$

Extended object σ would be expected to decrease faster w/ θ.

From Text by Roy & Nigam

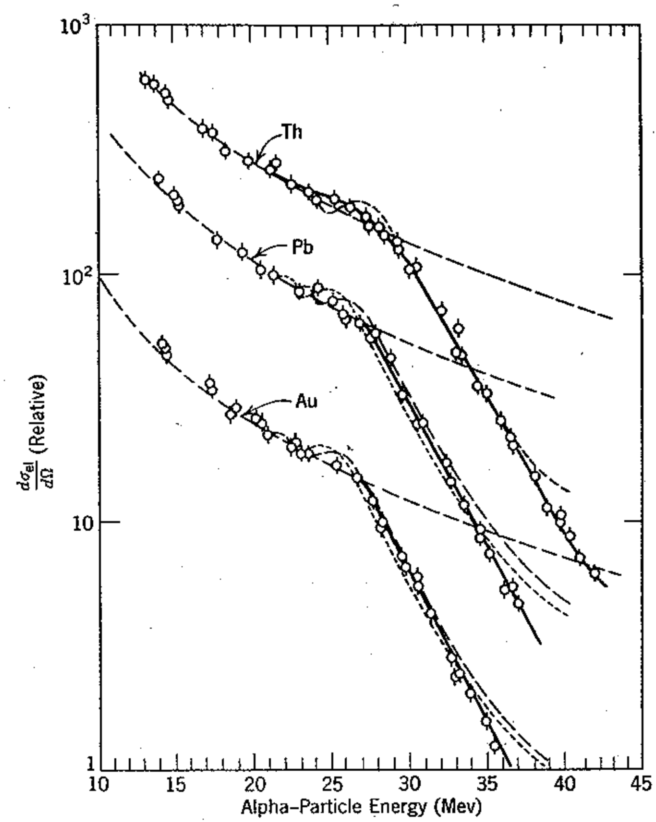


Fig. 2-1 The broad dashed curve gives the Coulomb cross section, and the solid curve represents the experimental data of Farwell and Wegner[4] for Au, Pb, and Th. For Au, the finer theoretical curve corresponds to $R = 10.58 \times 10^{-13}$ cm and the coarser curve to $R = 10.3 \times 10^{-13}$ cm. For Pb, the finer curve corresponds to $R = 10.87 \times 10^{-13}$ cm and the coarser to $R = 10.42 \times 10^{-13}$ cm. For Th, the dashed curve corresponds to $R = 11.01 \times 10^{-13}$ cm (Eisberg and Porter[1]).

--- Rutherford scattering

Distance scale probed by α:

Classical minimum distance r_{min}:

$$E_\alpha = \frac{ZZ'e^2}{r_{min}}$$

$$\frac{e^2}{\hbar c} = \alpha \cong \frac{1}{137}$$ fine structure constant

take $E_\alpha = 4$ MeV on $^{207}_{82}Pb$

$$r_{min} = \frac{2(82)}{137} \frac{200\,MeV}{4\,MeV} = \frac{8200}{137}\,fm$$

$$= 60\,fm$$

nucleon size $r_n \cong 1.2\,fm\,A^{1/3}$

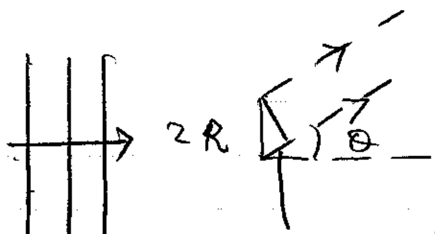
$$= 1.2\,(207)^{1/3}\,fm = 7\,fm$$

$r_{min} >> r_{pb}$.

for $E = 25$ MeV, $r_{min} \cong 10\,fm$. Begin to probe nucleon size.

low energy
elastic π - nucleus scattering: $\frac{d\sigma}{d\Omega}(\theta)$

shows structure that can be interpreted as wave scattering by absorptive sphere:



$$\Delta = 2R\,\theta_n = n\lambda = n\frac{h}{p}$$

$$\lambda = \frac{h}{p}$$

<u>note</u>: first dip for π - Pb scattering is slightly above 50°

Mott cross section

Relativistic ($v \cong 1$) e^- elastic scattering by spinless, point-like nucleus of mass M, charge Ze:

$$\frac{d\sigma}{d\Omega}^{Mott} = \frac{(Z\alpha\hbar c)^2}{4E_0^2 \sin^4\frac{\theta}{2}} \left(\frac{E}{E_0}\right) \cos^2\frac{\theta}{2}$$

ultra-relativistic $E^2 = p^2$

recoil $v = 1$

e^- spin

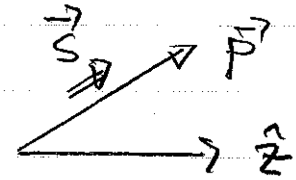
In ultra-relativistic limit, helicity is conserved.

$$H = \left\langle \frac{\vec{S} \cdot \vec{p}}{|\vec{S}| \cdot |\vec{p}|} \right\rangle = \pm 1$$

$\overset{\vec{S}}{\Rightarrow}$ $\underset{\vec{p}}{\longrightarrow}$ "right handed"

$\overset{\vec{S}}{\Leftarrow}$ $\underset{\vec{p}}{\longrightarrow}$ "left handed"

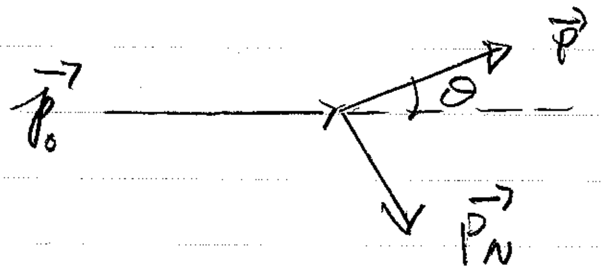
Incoming wave function with $\vec{p} = p\hat{z}$ has $L_z = 0$. Conservation of angular momentum requires that $\langle \vec{S} \cdot \hat{z} \rangle$ does not change



$$|s\rangle_{\vec{p}} = |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle_p$$

Amplitude $\;{}_z\langle \tfrac{1}{2}, \pm\tfrac{1}{2} | \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle_p = \cos\frac{\theta}{2}$

Take nuclear recoil into account



$$\bar{P} = P_0(1, \hat{z}) \; ; \quad \bar{p} = p(1, \hat{r}) \; ; \quad \bar{p}_0 \cdot \bar{p}_0 = 0 \; ; \; \bar{p}\cdot\bar{p} = 0$$

4 momentum transfer :

$$-q^2 = -(\bar{p}_0 - \bar{p})^2 = 2p_0 p\,(1 - \cos\theta)$$

$$= 4E_0 E \sin^2\frac{\theta}{2}$$

Energy conservation determines $E(\theta)$:

$$E_0 + m_N = E + E_N$$

$$(E_0 + m_N - E)^2 = E_N^2 = m_N^2 + |\vec{P}_N|^2$$

$$|P_N|^2 = |\vec{p}_0 - \vec{p}|^2 = p_0^2 + p^2 - 2p_0 p \cos\theta$$

$$= E_0^2 + E^2 - 2E_0 E \cos\theta$$

$$= (E_0 - E)^2 + 4E_0 E \sin^2\theta/2$$

$$\cancel{(E_0 - E)^2} + 2m_N(E_0 - E) = \cancel{(E_0 - E)^2} + 4E_0 E \sin^2\theta/2$$

$$\left(\frac{1}{E} - \frac{1}{E_0}\right) = \frac{2\sin^2\theta/2}{m_N}$$

$$E^{-1} = E_0^{-1}\left(1 + \frac{2E_0}{m_N}\sin^2\frac{\theta}{2}\right)$$

Nucleon Form factor:

$$\mathcal{M}^{(1)}_{fi} = \int \psi_f^* e \phi(\vec{x}) \psi_i \, d^3x$$

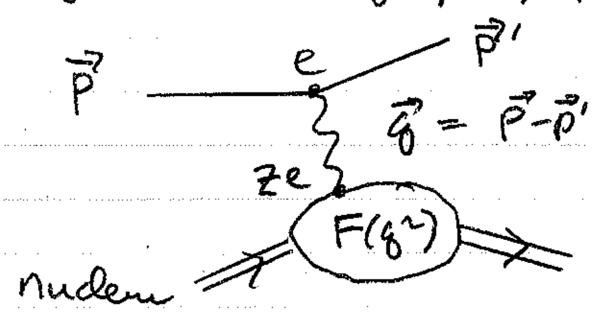
$$\nabla^2 \phi(\vec{x}) = -4\pi Z e \, \rho(\vec{x}) \quad ; \quad \int_0^\infty \rho(\vec{x}) \, d^3x = 1$$

Use Green's theorem: $\int (u \nabla^2 v - v \nabla^2 u) \, d^3x = 0$

to get

$$\mathcal{M}^{(1)}_{fi} = \frac{-4\pi Z e^2}{V} \frac{\hbar^2}{q^2} \underbrace{\int e^{i \vec{q} \cdot \vec{x}/\hbar} \rho(x) \, d^3x}_{\text{nuclear form factor } \equiv F(q^2)}$$

as Feynman-like graph, point charge $\rightarrow F(q^2)$



$$\propto \frac{Z e^2}{q^2} F(q^2)$$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}^{\text{Mott}} \left| F(q^2) \right|^2$$

For $f(x) = f_0 e^{-x/a}$

$$F(q^2) = \left(1 + \frac{-q^2}{a^2}\right)^{-1} \quad \text{"dipole"}$$

better approximation,

$$f(x) = f_0 \left[1 + \exp\left(\frac{r-c}{a}\right)\right]^{-1}$$

data given $c \cong 1.07 \text{ fm } A^{1/3}$; $a = 0.54 \text{ fm}$

see figure on next page.

Homogeneous charged sphere approximation,

$$r_{rms} = \sqrt{\langle r^2 \rangle} = 1.2 \text{ fm } A^{1/3}$$

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from Povh, Rith, Scholz, Zetsche

- Nuclei are not spheres with a sharply defined surface. In their interior, the charge density is nearly constant. At the surface the charge density falls off over a relatively large range. The radial charge distribution can be described to good approximation by a Fermi function with two parameters

$$\varrho(r) = \frac{\varrho(0)}{1 + \mathrm{e}^{(r-c)/a}} . \tag{5.52}$$

  This is shown in Fig. 5.8 for different nuclei.
- The constant c is the radius at which $\varrho(r)$ has decreased by one half. Empirically, for larger nuclei, c and a are measured to be:

$$c = 1.07\ \mathrm{fm} \cdot A^{1/3} , \qquad a = 0.54\ \mathrm{fm} . \tag{5.53}$$

- From this charge density, the mean square radius can be calculated. Approximately, for medium and heavy nuclei:

$$\langle r^2 \rangle^{1/2} = r_0 \cdot A^{1/3} \qquad \text{where} \quad r_0 = 0.94\ \mathrm{fm} . \tag{5.54}$$

  The nucleus is often approximated by a homogeneously charged sphere. The radius R of this sphere is then quoted as the nuclear radius. The following connection exists between this radius and the mean square radius:

$$R^2 = \frac{5}{3} \langle r^2 \rangle . \tag{5.55}$$

  Quantitatively we have:

$$R = 1.21 \cdot A^{1/3}\ \mathrm{fm} . \tag{5.56}$$

  This definition of the radius is used in the mass formula (2.8).
- The surface thickness t is defined as the thickness of the layer over which the charge density drops from 90 % to 10 % of its maximal value:

$$t = r_{(\varrho/\varrho_0 = 0.1)} - r_{(\varrho/\varrho_0 = 0.9)} . \tag{5.57}$$

  Its value is roughly the same for all heavy nuclei, namely:

$$t = 2a \cdot \ln 9 \approx 2.40\ \mathrm{fm} . \tag{5.58}$$