Phys 450

spring 2021

Lec #3 Symmetries I

1) Symmetry & conservation laws

Noether's theorem (field theory)

Invariance of action $S = \int \mathcal{L}\, dt$ under transformation gives conservation law

	Symmetry	conserved quantity
spacetime	time translation	E
	space translation	$\vec{p}$
	Rotation	$\vec{J}$ (angular momentum)
internal	Gauge	Q charge

transformations are <u>continuous</u>

mathematically Lie groups (pronounced "Lee")

<u>Group</u> set of elements $G \equiv \{g_i\}$ with group multiplication defined such that

1) Closure $\quad g_i \cdot g_j = g_k$

2) identity exists $\quad 1 \cdot g_i = g_i$

3) inverse exists $\quad g_i^{-1} g_i = 1$

4) associativity $\quad g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k$

for Euclidean rotations, elements 3×3 orthogonal matrices with matrix multiplication recall angular momentum in Q.M.

$\hat{L}^2 = |\hat{\vec{L}}|^2$ commutes with $\hat{L}_z$ eigenstates

$$\hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle \quad l = 1,2,3 \text{ integer}$$

$$\hat{L}_z |l,m\rangle = \hbar m |l,m\rangle \quad -l \le m \le l$$

m integer

$$\left[\frac{\hat{L}_x}{\hbar}, \frac{\hat{L}_y}{\hbar}\right] = i\frac{\hat{L}_z}{\hbar} \text{ and cyclic or}$$

$$\left[\frac{\hat{L}_i}{\hbar}, \frac{\hat{L}_j}{\hbar}\right] = i\sum_k \varepsilon_{ijk}\frac{\hat{L}_k}{\hbar}$$

structure constants of SU(2)

rotations are 3x3 matrices acting on Euclidean vector

Lowest dimensional representation of SU(2) is "spinor" acting on a spinor $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ 2 component, complex

group elements 2x2 unitary matrices.

$$\hat{U}(\theta) = \exp\left(-i\,\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)$$

defined by Taylor expansion

3 parameters

3 generators are Pauli matrices (Hermitian)

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Hermitian generator gives unitarity

$$\bar{U}^{+}\bar{U} = \exp\left(i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}^{+}\right)\exp\left(-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right) = \bar{1}$$

$\frac{\bar{\sigma}_i}{2}$ are called generators

* generator of $SU(N) = N^2 - 1$

Satisfy $SU(2)$ commutation relation

$$\left[\frac{\bar{\sigma}_i}{2}, \frac{\bar{\sigma}_j}{2}\right] = i\sum_k \varepsilon_{ijk}\frac{\bar{\sigma}_k}{2}$$

Since spin $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ operators satisfy same $SU(2)$ commutation relations, we have

$[\hat{S}^2, \hat{S}_z] = 0$ and spin eigenstates

$$\hat{S}^2 \left|\tfrac{1}{2}, m\right\rangle = \hbar\, \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)$$

$$\hat{S}_z \left|\tfrac{1}{2}, m\right\rangle = \pm m\hbar \left|\tfrac{1}{2}, m\right\rangle \quad -\tfrac{1}{2} \le m \le \tfrac{1}{2} \text{ in unit steps}$$

$$m = \pm\tfrac{1}{2}$$

<u>Product States</u> "addition of angular momentum"

decompose product of states into sum of multiplets that do not mix under group transformation, = Irreducible Representations "irrep"

rules for SU(2) are very simple

product $j_1 \times j_2$ = irreps $|j_1 - j_2| \leq j \leq j_1 + j_2$

in unit steps.

in this decomposition we are just rearranging the states, so total number is the same.

<u>example</u> $(j=1) \times (j=1)$

$$\underset{\sim}{1} \otimes \underset{\sim}{1} = \underset{\sim}{0} \oplus \underset{\sim}{1} \oplus \underset{\sim}{2}$$

(singlet: $\underset{\sim}{0}$; vector: $\underset{\sim}{1}$)

mult: $= 2j+1$

$$3 \times 3 = 9 = \underset{\sim}{1} \oplus \underset{\sim}{3} \oplus \underset{\sim}{5}$$

(note: mathematicians denote representation by multiplicity as

$$\underset{\sim}{3} \otimes \underset{\sim}{3} = \underset{\sim}{1} + \underset{\sim}{3} + \underset{\sim}{5}$$)

we denote by highest eigenvalue of $\hat{J}_z$

What are the direct sums of $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ in each irrep? We can look these up in Clebsch-Gordon Table (see link on web page)

* Flavor Symmetries: Global symmetry
(global means does not depend on spacetime point)

Start with isospin.
Nuclear force is same for p, n "charge independent"
symmetry can be made explicit in terms of $SU(2)$ isospin:

$$|p\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \qquad |n\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

However, nuclear (pion) force is spin dependent

$\uparrow\uparrow$ spin is slightly stronger attracted than
$\uparrow\downarrow$ opposite spin

The deuteron $d = pn$ bound state

Spin $j=1$, <u>weakly</u> <u>bound</u> ~ 2 MeV

Pauli exclusion principle for "identical" nucleons (under the strong-nuclear force) must be antisymmetric under exchange of nucleons.

$|d\rangle = |\text{space}\rangle |\text{spin}\rangle |\text{isospin}\rangle$

for pn ground state $l=0$, $|\text{space}\rangle$ symmetric

Two possibilities are

$|d'\rangle = |\text{spin}\rangle_A |\text{isospin}\rangle_S$

$|d\rangle = |\text{spin}\rangle_S |\text{isospin}\rangle_A$

A ≡ antisymmetric
S ≡ symmetric
under nucleon exchange

$|\text{spin}\rangle_A = \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$ anti-aligned spins more <u>weakly</u> bound

$|\text{spin}\rangle_S = \begin{cases} \uparrow_1\uparrow_2 \\ \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2) \\ \downarrow_1\downarrow_2 \end{cases}$ aligned spins more strongly bound

$|I=1\rangle_S = \begin{cases} p_1 p_2 \\ \frac{1}{\sqrt{2}}(p_1 n_2 + n_1 p_2) \\ n_1 n_2 \end{cases}$ $|I=0\rangle_A = \frac{1}{\sqrt{2}}(p_1 n_2 - n_1 p_2)$

Since d is weakly bound, d' is unbound, <u>di-neutron does not exist.</u> (di-proton would not be expected to exist because Coulomb repulsion breaks isospin symmetry)