Phys 450 Spring 2021

Lec 7: Dirac Equation

In Q.M. $\hat{\vec{p}} = i\vec{\nabla}$ $(\hbar = 1)$

Dirac looked for matrices $\vec{\alpha}, \beta$ satisfying

$$\hat{H}\psi = (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m)\psi$$

and $\hat{H}^2\psi = (|\hat{\vec{p}}|^2 + m^2)\psi$ relativistic E-$\vec{p}$ relation

$$\hat{H}^2 = \underbrace{\alpha_i^2}_{1} p_i^2 + \underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{0} p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0} p_i m + \underbrace{\beta^2}_{1} m^2$$

repeated indices are summed

lowest dim. is 4 with one representation

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} ; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

or in covariant form

$$\gamma^\mu = (\beta, \beta\vec{\alpha}) \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \underline{\text{anti-commutator}} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\beta\hat{H}\psi = (\beta\vec{\alpha}\cdot\vec{p} + \beta^2 m)\psi$$

with $\hat{H} = i\frac{\partial}{\partial t}$ this becomes

$$\beta i\frac{\partial}{\partial t}\psi = \beta\vec{\alpha}\cdot\frac{1}{i}\vec{\nabla}\psi + m\psi$$

$$\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = 0$$

Covariant Dirac equation

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \text{Dirac spinor}$$

Adjoint equation:

$$-i\frac{\partial \psi^+}{\partial t}\gamma^0 - i\frac{\partial \psi^+}{\partial x^k}\underbrace{(\gamma^k)^+}_{-\gamma^k} - m\psi^+ = 0$$

multiply by γ^0 on the right and define

$$\bar{\Psi} \equiv \psi^+\gamma^0$$

$$\gamma^k\gamma^0 = -\gamma^0\gamma^k \qquad \text{multiply by } (-1) \text{ to get}$$

$$i\frac{\partial}{\partial t}\bar{\Psi}\gamma^0 + i\frac{\partial\bar{\Psi}}{\partial x^k}\gamma^k + m\bar{\Psi} = 0$$

$$\boxed{i\,\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0}$$

<u>Conserved current</u> -

$$\bar{\Psi}(i\gamma^\mu\partial_\mu)\psi = m\bar{\Psi}\psi \qquad \text{Dirac}$$

$$(i\,\partial_\mu\bar{\Psi}\gamma^\mu)\psi = -m\bar{\Psi}\psi \qquad \text{Conjugate}$$

add

$$\partial_\mu(\bar{\Psi}\gamma^\mu\psi) = 0$$

Conserved electric charge current is

$$j_\mu = -e\,\bar{\Psi}\gamma^\mu\psi \qquad \text{a 4-vector}$$

Solutions to Dirac

In e^- rest frame $\vec{p} = 0$

$$i\gamma^0 \frac{\partial \psi}{\partial t} = m\psi$$

let $\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ upper 2

$\psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$ lower 2

$$i \frac{\partial \psi_A}{\partial t} = m\psi_A \qquad \psi_A(t) = e^{-imt} \qquad \text{positive energy}$$

$$-i \frac{\partial \psi_B}{\partial t} = m\psi_A \qquad \psi_B(t) = e^{+imt} \qquad \text{negative energy}$$

4 eigenvectors:

$\psi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\psi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ positive energy

E = +/-sqrt(p^2+m^2)

$\psi^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\psi^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ negative energy

interpreted as positron

For $\vec{p}$ non-zero; plane wave solution

$$\psi = a\, e^{-i x \cdot p}\, u(\vec{p}) \qquad \text{where } x \cdot p = Et - \vec{p} \cdot \vec{x}$$

↑ 4-component

$$\partial_\mu \psi = -i p_\mu \psi \qquad \text{4-momentum eigenstate}$$

$$(\gamma^\mu p_\mu - m)\, u(\vec{p}) = 0 \qquad \text{equation for } u(\vec{p})$$

non-zero $\vec{p}$ plane wave solution continued

$$\gamma^\mu p_\mu = \gamma^0 E - \vec{\gamma}\cdot\vec{p}$$

$$= \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}$$

with $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$\begin{pmatrix} (E-m)\,u_A - \vec{p}\cdot\vec{\sigma}\; u_B \\ \vec{p}\cdot\vec{\sigma}\; u_A - (E+m)\, u_B \end{pmatrix} = 0$$

or $u_A = \dfrac{\vec{p}\cdot\vec{\sigma}}{E-m}\, u_B$; $u_B = \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\, u_A$

$$u_A = \frac{1}{E^2-m^2}\,(\vec{p}\cdot\vec{\sigma})^2\, u_A = \left(\frac{p^2}{E^2-m^2}\right) u_A$$

recall $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

gives $(E^2 - m^2 - p^2)\, u_A = 0$

two roots $E_\pm = \pm\sqrt{p^2+m^2}$

negative energy root must be re-interpreted as positron write as

$$U^{(1)} = N\left(1, 0, \frac{P_z}{E+m}, \frac{P_x + iP_y}{E+m}\right)^T$$ ← transpose

spin up & small components

$$U^{(2)} = N\left(0, 1, \frac{P_x - iP_y}{E+m}, \frac{-P_z}{E+m}\right)^T$$ spin down & small components

Choose Normalization $U^\dagger U = 2|E|$

$$2E = N^2\left(1 + \frac{P^2}{(E+m)^2}\right) = \frac{N^2\, 2E(E+m)}{(E+m)^2}$$

$$N = \sqrt{E+m}$$

Plane wave:

$$\psi(\vec{x},t) = a\, e^{-i(Et - \vec{p}\cdot\vec{x})}\, u^{(i)}(E,\vec{p}) \qquad i=1,2 \quad \text{electron}$$

for positron, $E = -|E|$, $\vec{P} = -\vec{p}$

$$= a\, e^{-i(-|E|t - (-\vec{p}\cdot\vec{x}))}\, U^{(i)}(-|E|, -\vec{p}) \qquad i=3,4$$

write $|E| = E$ as always positive

$$\psi(\vec{x},t) = a\, e^{i(Et - \vec{p}\cdot\vec{x})}\, u^{(i)}(-E, -\vec{p}) \qquad i=3,4$$

solutions for $u^{(i)}$ $i=1,2$

$$U^{(1)} = N\left(1, 0, \frac{p_z}{E+m}, \frac{p_x + i p_y}{E+m}\right)^T$$ ← transpose

spin-up & small

$$U^{(2)} = N\left(0, 1, \frac{p_x - i p_y}{E+m}, \frac{-p_z}{E+m}\right)^T$$ spin-down & small

Choose normilization $u^{+}u = 2|E|$

$$2E = N^2\left(1 + \frac{p^2}{(E+m)^2}\right) = \frac{N^2(2E)(E+m)}{(E+m)^2}$$

$$N = \sqrt{E+m}$$

plane waves are $i=1,2$

$$\psi(\vec{x}, t) = a\, e^{-i(Et - \vec{p}\cdot\vec{x})}\, U^{(i)}(E, p)$$

<u>Negative energy solutions must re-interpret as positive energy positrons,</u>

$$\text{phase} = Et - \vec{p}\cdot\vec{x} = (-E)t - \vec{p}\cdot\vec{x}$$

$$= -\left(Et - (-\vec{p}\cdot\vec{x})\right)$$

take $\vec{p} \to -\vec{p}$

$$\text{phase} = -\left(Et - \vec{p}\cdot\vec{x}\right) = -(p \cdot x)$$ ← $p^0 = E$

$\psi = a e^{+i(Et - \vec{p}\cdot\vec{x})}$ $\quad u^{(i)}(-E, -\vec{p}) \quad i=3,4$

define

$v^{(1)} \equiv u^{(4)}(-E, -\vec{p}) = N\left(\frac{P_x - iP_y}{E+m}, \frac{-P_z}{E+m}, 0, 1\right)^T$

$v^{(2)} \equiv -u^{(3)}(-E, -\vec{p}) = -N\left(\frac{P_z}{E+m}, \frac{P_x + iP_y}{E+m}, 1, 0\right)^T$

You will relate $u^{(i)}$ to $v^{(i)}$ with

charge conjugation $\quad \psi_c = i\gamma^2\psi^*$

<u>Lorentz transformation of Dirac wave function</u>

You will show the operator

that takes $\hat{S}_x \psi(x) = \psi'(x')$

where $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ for a boost

along the x-direction $v = \tanh\theta$

$$\hat{S}_x = \cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}\gamma^0\gamma^1$$

and $\hat{S}_x^{-1} = \cosh\theta/2 + \sinh\frac{\theta}{2}\gamma^0\gamma^1$

also $\hat{S}_x^{\dagger} = \cosh\theta/2 - \sinh\theta/2 \underbrace{(\gamma^0\gamma^1)^{\dagger}}_{\gamma^{1\dagger}\gamma^0 = -\gamma^1\gamma^0 = +\gamma^0\gamma^1}$

$$\boxed{\hat{S}_x^{\dagger} = \hat{S}_x}$$

Dirac Adjoint

Some useful γ matrix identities :

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\gamma^{0\dagger} = \gamma^0 \qquad (\gamma^i)^\dagger = -\gamma^i$$

$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$

Dirac adjoint $\bar\psi \equiv \psi^\dagger \gamma^0$

$\bar\psi \psi$ is Lorentz invariant.

$$\hat S_x \gamma^0 = (\underbrace{c}_{\cosh\theta/2} - \overbrace{s}^{\sinh\theta/2}\gamma^0\gamma^1)\gamma^0 = \gamma^0(c - s\gamma^1\gamma^0) = \gamma^0(c + s\gamma^0\gamma^1)$$
$$= \gamma^0 \hat S_x^{-1}$$

$$\bar\psi'(x')\psi'(x') = \psi'^\dagger(x')\gamma^0\psi'(x')$$
$$= (\hat S_x \psi(x))^\dagger \gamma^0 (\hat S_x \psi(x))$$
$$= \psi^\dagger(x)\, \hat S_x \gamma^0 \hat S_x \psi(x)$$
$$= \psi^\dagger(x)\gamma^0 (\hat S_x^{-1}\hat S_x)\psi(x) = \bar\psi(x)\psi(x)$$

Fun with spinors

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix} \qquad \chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad N = \sqrt{E+m}$$

$$u^{(1)} = N \begin{pmatrix} \chi_+ \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_+ \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} \chi_- \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_- \end{pmatrix}$$

$$v^{(1)} = N \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_- \\ \chi_- \end{pmatrix} \qquad v^{(2)} = -N \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_+ \\ \chi_+ \end{pmatrix}$$

Dirac equation:

$$\gamma^\mu p_\mu = \gamma^0 E - \vec{\gamma}\cdot\vec{p} = \begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}$$

$$\gamma^\mu p_\mu u^{(1)} = N \begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix} \begin{pmatrix} \chi_+ \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_+ \end{pmatrix}$$

$$= N \begin{pmatrix} E\chi^+ - \dfrac{(\vec{\sigma}\cdot\vec{p})^2}{E+m}\chi^+ \\ \vec{p}\cdot\vec{\sigma}\,\chi^+ - \dfrac{E(\vec{\sigma}\cdot\vec{p})}{E+m}\chi^+ \end{pmatrix}$$

$(\vec{\sigma}\cdot\vec{p})^2 = p^2$

$\dfrac{p^2}{E+m} = \dfrac{E^2-m^2}{E+m} = E-m$

$1 - \dfrac{E}{E+m} = m$

$$= N \begin{pmatrix} m\chi^+ \\ \left(\dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) m\chi^+ \end{pmatrix} = m\, u^{(1)}$$

Note that for antiparticles, spin and momentum are reversed. The helicity is unchanged. (In rest frame, absence of electron with spin up in Dirac "sea" (hole theory) is equal to the presence of a positron with spin down.

Bilinear covariants

$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\bar{\psi}\psi$ scalar

$\bar{\psi}\gamma^5\psi$ pseudo scalar

$\bar{\psi}\gamma^\mu\psi$ vector

$\bar{\psi}\gamma^\mu\gamma^5\psi$ – pseudo vector

$\bar{\psi}\sigma^{\mu\nu}\psi$ antisymmetric tensor

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$$

example:

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \left(\hat{S}_x\psi(x)\right)^\dagger \gamma^0\gamma^\mu \hat{S}_x\psi(x)$$

$$= \psi^\dagger \underbrace{\hat{S}_x}\gamma^0\gamma^\mu \hat{S}_x \psi(x) = \bar{\psi}(x) S_x^{-1}\gamma^\mu S_x \psi(x)$$

$$\hat{S}_x\gamma^0 = \gamma^0\hat{S}_x^{-1}$$

you will show $\hat{S}_x^{-1}\gamma^\mu\hat{S}_x = \Lambda^\mu{}_\nu\gamma^\nu$

so $\bar{\psi}\gamma^\mu\psi$ transforms as a 4-vector

Hence, the electric current is

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi$$