Phys 480 Spring 2021

Lecture 8 Photon, QED

The electron current in QED

$$j^\mu = -e\,\bar{\psi}\gamma^\mu\psi$$

couples to the photon 4-potential

$$A^\mu = (V, \vec{A})$$

related to EM fields,
$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}V - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

Maxwell's Equations can be written in terms of covariant Maxwell field strength antisymmetric tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

and 4-current $j^\mu = (\rho, \vec{J})$

Maxwell's equations are then

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

We can write $F^{\mu\nu}$ as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

from which anti-symmetry is self evident, following from $\partial^\mu \partial^\nu = \partial^\nu \partial^\mu$.

$$\partial_\nu \left[\partial_\mu F^{\mu\nu} \right] - \frac{4\pi}{c} \partial_\nu J^\nu = 0$$

this is the continuity equation, conservation of electric charge,

$$\partial_\nu J^\nu = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

<u>aside</u> $\quad x^\mu = (t, \vec{x}) \qquad x \cdot x = t^2 - \vec{x} \cdot \vec{x}$

$$x_\mu = (t, -\vec{x})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \qquad \partial^2 = \Box = \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

4-momentum operator $\hat{p}^\mu = i\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

$$\vec{P}_\mu = i\left(\frac{\partial}{\partial t}, +\vec{\nabla}\right)$$

QED interaction is

$$j_\mu A^\mu = -e \bar{\psi} \gamma_\mu \psi A^\mu$$

electron couples to EM 4-potential

Gauge Invariance

The transformation $A'_\mu = A_\mu + \partial_\mu \lambda$

with scalar function $\lambda(t, \vec{x})$ arbitrary

leaves $F^{\mu\nu}$ unchanged.

$$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = F^{\mu\nu} + \partial^\mu \partial^\nu \lambda - \partial^\nu \partial^\mu \lambda = F^{\mu\nu}$$

This allows the constraint

$$\partial_\mu A^\mu = 0 \qquad \text{Lorentz Condition}$$

then Maxwell's equations are

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu + \partial^\nu \partial_\mu A^\mu - \partial_\mu \partial^\mu A^\nu = \partial^2 A^\nu$$

$$\Rightarrow \partial^2 A^\nu = \frac{4\pi}{c} J^\nu$$

Finally, completely fix Gauge as <u>Coulomb Gauge</u>

$$A^0 = 0 \quad \vec{\nabla} \cdot \vec{A} = 0$$

Gauge Invariance in Lagrangian Field theory (ch 11)

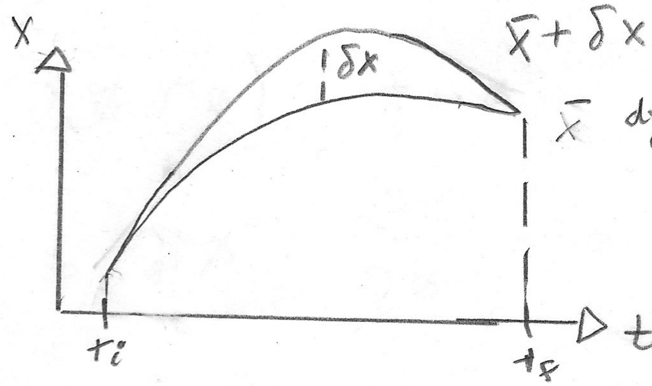
In classical mechanics

$$\mathcal{L}(x,\dot{x}) = \tfrac{1}{2} m\dot{x}^2 + V(x)$$

Equations of motion follow from principle of least action. True path $\bar{x}$, $\dot{\bar{x}}$ minimizes action

$$\int \mathcal{L}(\bar{x},\dot{\bar{x}})\,dt = \text{extremal (min or max)}$$

Variation $\delta x = 0$ @ t_i, t_f



$\bar{x} + \delta x(t)$ varied path

$\bar{x}$ dynamical path

$$\delta\dot{x} = \frac{d}{dt}\delta x$$

note $dx = x(t+dt) - x(t)$ compares x at different times

variation δx is difference in paths at same time

$$\delta\mathcal{L} = \mathcal{L}(\bar{x}+\delta x, \dot{\bar{x}}+\delta\dot{x}) - \mathcal{L}(\bar{x},\dot{\bar{x}})$$

$$= \frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta\dot{x}$$

$$\delta\int\mathcal{L}\,dt = 0 = \int\left[\frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial\dot{x}}\frac{d}{dt}\delta(x)\right]dt$$

$$= \int\left(\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)\overbrace{\delta x}^{\text{arbitrary}}\,dt = 0$$

integrate by parts

given Euler Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$$

for example $\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x)$

$$m\ddot{x} + \frac{\partial V}{\partial x} = 0 \qquad m\ddot{x} = -\frac{\partial V}{\partial x} \quad \checkmark$$

Scalar field

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi\right) - \frac{1}{2} m \phi^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m\phi$$

$$\partial_\mu \partial^\mu \phi + m\phi = 0 \qquad \text{Klein Gordon}$$

Dirac Field

$$\mathcal{L} = i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \bar{\Psi})} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = i \gamma^\mu \partial_\mu \Psi - m\Psi$$

so $i \gamma^\mu \partial_\mu \Psi - m\Psi = 0$ Dirac

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi)} = i\,\bar{\psi}\gamma^\mu \qquad \frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi}$$

$$i\,\partial^\mu(\bar{\psi}\gamma_\mu) + m\bar{\psi} = 0 \qquad \text{adjoint equation}$$

Suppose we require $\mathcal{L}$ be invariant under local phase transformation

$$\psi \to \psi' = e^{-ie\lambda(x)}\psi \qquad \begin{array}{l} e \text{ electric charge} \\ \lambda(t,\vec{x}) \text{ arbitrary} \end{array}$$

$$m\bar{\psi}'\psi' = m\bar{\psi}\psi \quad \text{invariant but}$$

$$i\bar{\psi}'\gamma^\mu\partial_\mu\psi' = i e^{ie\lambda}\gamma^\mu\partial_\mu(e^{-ie\lambda}\psi)$$

$$= i e^{ie\lambda}\bar{\psi}\gamma^\mu(-ie\partial_\mu\lambda\, e^{-ie\lambda}\psi + e^{-ie\lambda}\partial_\mu\psi)$$

$$= i\bar{\psi}\gamma^\mu(\partial_\mu - ie\,\partial_\mu\lambda)\psi \qquad \text{not invariant}$$

but $i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi$ <u>is</u> invariant under combined

$$\left.\begin{array}{l} \psi' = e^{-ie\lambda}\psi \\ A'_\mu = A_\mu + \partial_\mu\lambda \end{array}\right\} \text{Gauge transformation}$$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \underline{e\bar{\psi}\gamma^\mu\psi A_\mu} - m\bar{\psi}\psi \quad \text{is}$$

locally gauge invariant.

Coupling of e to γ determined

Scattering theory

"Far" from interaction region, particles are free and can be described by free-particle plane waves.

$$\psi_{e^-} = a\, e^{-i\, x \cdot p}\, u^{(s)}(p) \qquad \text{electron}$$

$$\psi_{e^+} = a\, e^{+i\, x \cdot p}\, v^{(s)}(p) \qquad \text{positron}$$

normalized as $\bar{u}u = +2mc$ $\quad \bar{v}v = -2mc$

complete as (you will show)

$$\sum_s u^s \bar{u}^s = \gamma^\mu p_\mu + m$$

$$\sum_s v^s \bar{v}^s = \gamma^\mu p_\mu - m$$

sum over polarizations simplify

photons

$$A^\mu(x) = a\, \varepsilon^\mu_{(s)}\, e^{-ip \cdot x}$$

polarization vectors $\varepsilon^\mu_{(s)}$ $\quad s = 1, 2$

photon polarization

In Coulomb gauge $\varepsilon^0 = 0$ $\vec{\varepsilon}\cdot\vec{p} = 0$

Two orthogonal vectors

$$\vec{\varepsilon}_i \cdot \vec{\varepsilon}_j = 0 \quad \text{with } i,j = 1,2$$

not necessarily real, with $\vec{p} = p_z$

linear $\varepsilon_1 = (1,0,0)$ $\varepsilon_2 = (0,1,0)$

circular $\varepsilon_\pm = \mp\frac{1}{\sqrt{2}}(1,\pm i,0)$ right, left

<u>Completeness:</u> (ex 7.25)

$$\sum_{s=1,2} (\varepsilon_{(s)i}\, \varepsilon^*_{sj}) = \delta_{ij} - \hat{p}_i \hat{p}_j$$

with $\vec{p} = \hat{p}_z$ $\hat{p}_i \hat{p}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij}$

for linear polarization

$$\varepsilon_{(1)i}\varepsilon_{(1)j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_i (1\,0\,0)_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$\varepsilon_{2i}^{*}\varepsilon_{2j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_i (0\,1\,0)_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

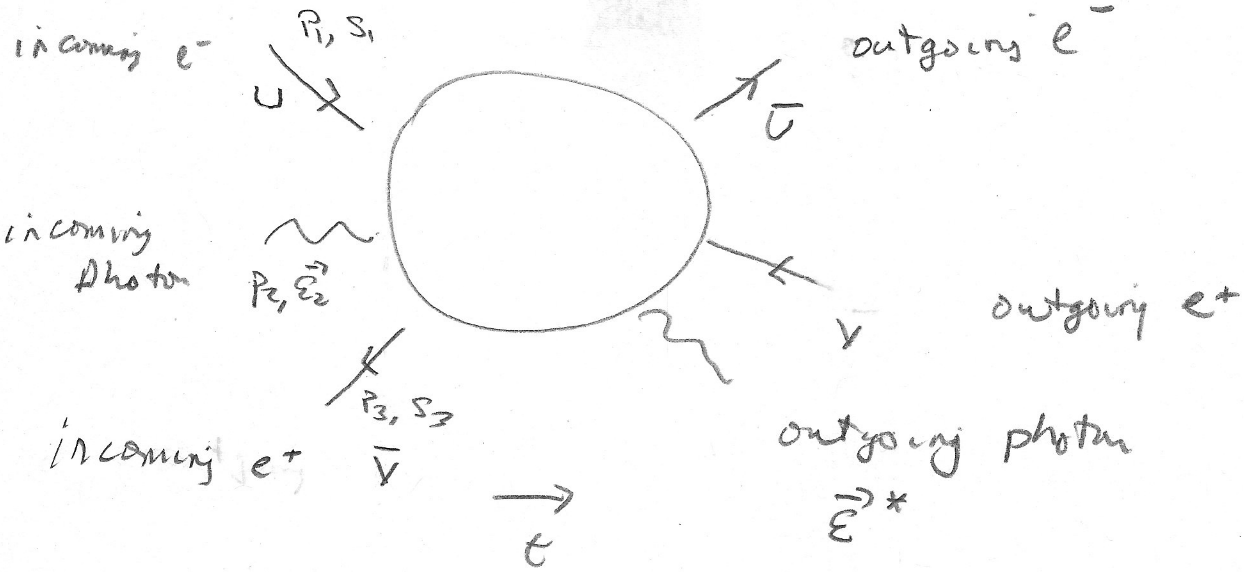
so $\sum_s \varepsilon_{(s)i}\varepsilon_{(s)j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} = \delta_{ij} - \hat{p}_i\hat{p}_j$

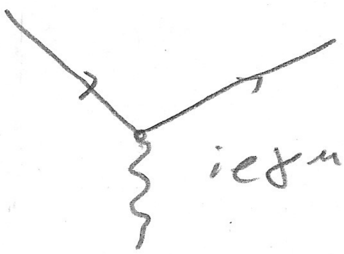
Skip all the quantum field theory, go directly to Feynman rules

p 4momentum

s spin index

$\vec{\epsilon}$ polarization vector





$ie\gamma^\mu$

Propagators

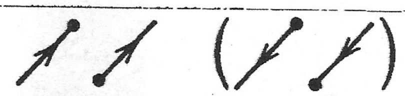
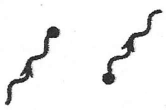
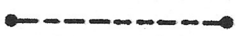
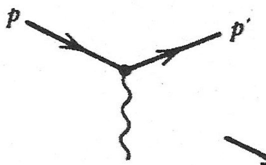
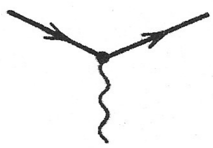
$$\frac{-i g_{\mu\nu}}{p^2}$$

$$\frac{i(\not{p}+m)}{p^2-m^2}$$

$\not{p} \equiv \gamma^\mu p_\mu$

From Halzen & Martin

TABLE 6.2
Feynman Rules for $-i\mathcal{M}$

		Multiplicative Factor
• **External Lines**		
Spin 0 boson (or antiboson)		1
Spin $\frac{1}{2}$ fermion (in, out)		$u, \bar{u}$
antifermion (in, out)		$\bar{v}, v$
Spin 1 photon (in, out)		$\varepsilon_\mu, \varepsilon_\mu^*$
• **Internal Lines—Propagators (need $+i\varepsilon$ prescription)**		
Spin 0 boson		$\dfrac{i}{p^2 - m^2}$
Spin $\frac{1}{2}$ fermion		$\dfrac{i(\not{p} + m)}{p^2 - m^2}$
Massive spin 1 boson		$\dfrac{-i(g_{\mu\nu} - p_\mu p_\nu/M^2)}{p^2 - M^2}$
Massless spin 1 photon (Feynman gauge)		$\dfrac{-ig_{\mu\nu}}{p^2}$
• **Vertex Factors**		
~~Photon—spin 0 (charge $-e$)~~		$\cancel{ie(p+p')^\mu}$
Photon—spin $\frac{1}{2}$ (charge $-e$)		$ie\gamma^\mu$

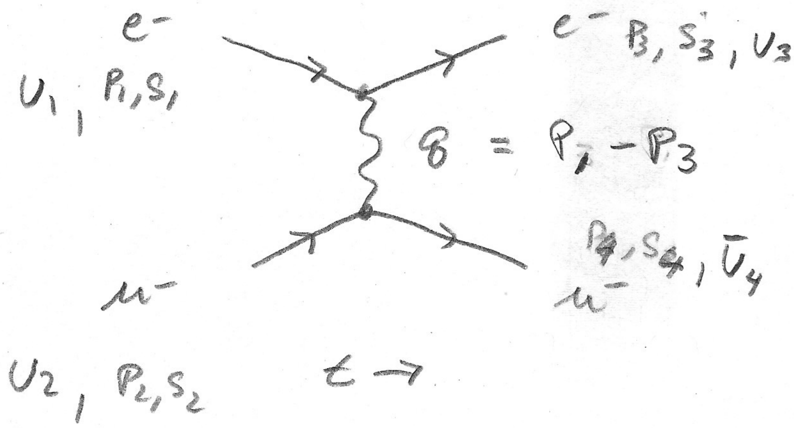
ignore spin case

Loops: $\int d^4k/(2\pi)^4$ over loop momentum; include -1 if fermion loop and take the trace of associated γ-matrices
Identical Fermions: -1 between diagrams which differ only in $e^- \leftrightarrow e^-$ or initial $e^- \leftrightarrow$ final e^+

Halzen & Martin $e^2 = 4\pi\hbar c\alpha$
Griffiths $g_e^2 = 4\pi\alpha$
$e^2 = \hbar c\alpha$

with $\hbar c = 1$
$e^2 = g_e^2 = 4\pi\alpha$

<u>example</u> $e^- \mu^- \to e^- \mu^-$



<u>note</u> - I draw time horizontal

4 momentum conserved at every vertex

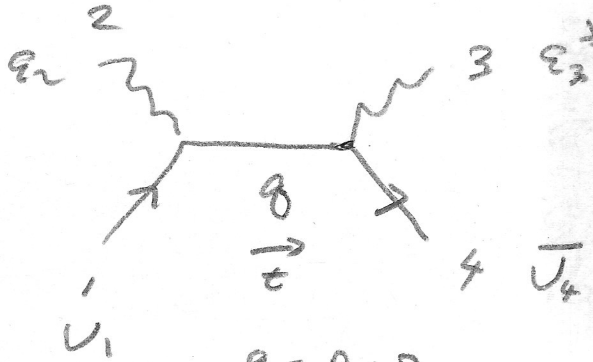
follow fermion lines <u>backwards</u>

$$-iM = \left[\bar{U}_4 \, (ie\gamma^\mu) \, U_2\right] \frac{-i g_{\mu\nu}}{q^2} \left[\bar{U}_3 \, ie\gamma^\nu \, U_1\right]$$

$$= ie^2 \left[\bar{U}_4 \, \gamma_\mu \, U_2\right]\left[\bar{U}_3 \, \gamma^\mu \, U_1\right]$$

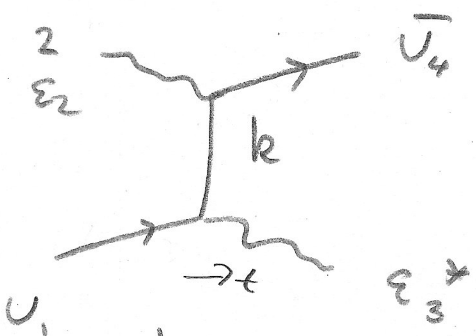
[ ] factors are numbers, not matrices, so order doesn't matter

<u>Example</u> Compton scattering $\gamma e^- \to \gamma e^-$



$q = p_1 + p_2$

M_A



$k = p_3 - p_1$

M_B

<u>notice where polarization vectors go.</u>

$$-iM_A = \bar{u}_4 \left(-ie\gamma^\mu \varepsilon^{*}_{3\mu}\right) \frac{i(\not{q}+m)}{q^2-m^2} (-ie)\gamma^\nu \varepsilon_{2\nu} u_1$$

$$= -ie^2 \, \bar{u}_4 \not{\varepsilon}_3^{*} \frac{\not{q}+m}{q^2-m^2} \not{\varepsilon}_2 u_1$$

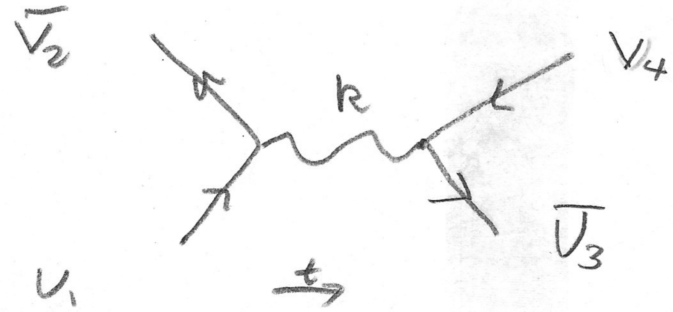
note

$\not{\varepsilon}_3^{*} \equiv \gamma^\mu \varepsilon_{3\mu}$

$$-iM_B = \bar{u}_4 \left(-ie\gamma_\mu \varepsilon_2^\mu\right) \frac{i(\not{k}+m)}{k^2-m^2} \left(-ie\gamma_\nu \varepsilon_3^{\nu *}\right) u_1$$

$$= -ie^2 \, \bar{u}_4 \not{\varepsilon}_2 \left(\frac{\not{k}+m}{k^2-m^2}\right)\left(\not{\varepsilon}_3^{*}\right) u_1$$

example $e^+e^- \to \mu^+\mu^-$



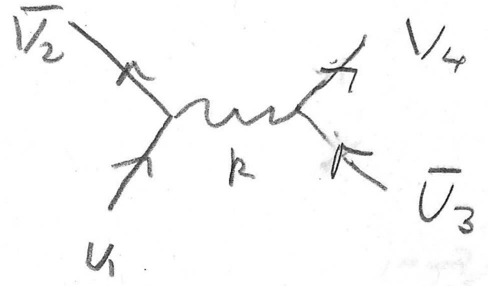
$k = p_1 + p_2$

$$-i\mathcal{M} = \bar{U}_3 (-ie\gamma_\mu) V_4 \left(\frac{-ig^{\mu\nu}}{k^2} \right) \bar{V}_2 (-ie\gamma_\nu) U_1$$

$$= +ie^2 (\bar{U}_3 \gamma^\mu V_4)(\bar{V}_2 \gamma_\mu U_1)$$

Next, Computing $|\mathcal{M}|^2$

Computing $|M|^2$ $e^+e^- \to \mu^+\mu^-$



$k = p_1 + p_2$

$$-iM = \frac{ie^2}{k^2}(\bar{U}_3 \gamma^\mu V_4)(\bar{V}_2 \gamma_\mu U_1) \equiv ie^2 A^\mu B_\mu$$

$$|A^\mu B_\mu|^2 = (A^\mu B_\mu)^*(A^\nu B_\nu) = A^{*\mu} A^\nu \; B^*_\mu B_\nu$$

$$|\bar{V}_2 \gamma_\mu U_1|^2 = (V_2^+ \gamma_0 \gamma_\mu U_1)^+ (\bar{V}_2 \gamma_\nu U_1)$$

$$= (U_1^+ \underbrace{\gamma_\mu^+ \gamma_0^+}_{-\gamma_\nu\gamma_0 = +\gamma_0\gamma_\nu} V_2)(\bar{V}_2 \gamma_\nu U_1)$$

$$= (\bar{U}_1 \gamma_\mu V_2)(\bar{V}_2 \gamma_\nu U_1)$$

$$\sum_{\text{Spins } 1,2} (\bar{U}_1 \gamma_\mu V_2)(\bar{V}_2 \gamma_\nu U_1) = \sum_{\text{Spins } 1} \bar{U}_1 \gamma_\mu \underbrace{\sum_{\text{Spin } 2} V_2 \bar{V}_2}_{(\not{p}_2 - m_e)} \gamma_\nu U_1$$

Completeness $\displaystyle\sum_{\text{spin}} U\bar{U} = \not{p} + m$

$$\sum_{\text{spin}} V\bar{V} = \not{p} - m$$

then $\sum_{spin} \bar{u}_1 \gamma_\mu \underbrace{(\not{p}_2 - m_e)\gamma_\nu}_{Q_{\mu\nu}} u_1$

$Q_{\mu\nu}$ 4 x 4 matrix in Dirac spinor indices

$$\sum_{i,j} \sum_{spin_1} \bar{u}_1^{\,i} Q_{\mu\nu}^{ij} u_1^j$$

$$= \sum_{i,j} Q_{\mu\nu}^{ij} \sum_{spin_1} \left(u_1^j \bar{u}_1^{\,i} \right)$$

$$= \sum_{i,j} Q_{\mu\nu}^{ij} \left(\not{p}_1 + m_e \right)^{ji} = tr\left(Q_{\mu\nu} (\not{p} + m) \right)$$

Dirac matrix trace

$$= tr\left(\gamma_\mu (\not{p}_2 - m_e) \gamma_\nu (\not{p}_1 + m_e) \right)$$

similarly

$$\sum_{spin\ 3,4} \left| \left(\bar{u}_3 \gamma^\mu v_4 \right) \right|^2 = \sum_{spin\ 3,4} \bar{v}_4 \gamma^\mu u_3 \bar{u}_3 \gamma^\nu v_4$$

$$= tr\left(\gamma^\mu (\not{p}_3 + m_\mu) \gamma^\nu (\not{p}_4 - m_\mu) \right)$$

Then $\langle |m|^2 \rangle = \frac{1}{4} \frac{e^4}{k^4} \alpha^{\mu\nu} \beta_{\mu\nu}$

$$\alpha^{\mu\nu} = tr\left(\gamma^\mu (\not{p}_3 + m_\mu) \gamma^\nu (\not{p}_4 - m_\mu)\right)$$

$$\beta_{\mu\nu} = tr\left(\gamma_\mu (\not{p}_2 - m_e) \gamma_\nu (\not{p}_1 + m_e)\right)$$

In high energy limit $m_e \simeq 0$, $m_\mu \simeq 0$.

$$\alpha^{\mu\nu} = tr\left(\gamma^\mu \not{p}_3 \gamma^\nu \not{p}_4\right)$$

$$= p_3^\lambda p_4^\rho \, tr\left(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\rho\right)$$

$$= p_3^\lambda p_4^\rho \, 4\left(g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\lambda\nu} - g^{\mu\nu} g^{\lambda\rho}\right)$$

$$= 4\left(p_3^\mu p_4^\nu + p_4^\mu p_3^\nu - g^{\mu\nu} p_3 \cdot p_4\right)$$

similarly

$$\beta_{\mu\nu} = 4\left(p_{2\mu} p_{1\nu} + p_{1\mu} p_{2\nu} - g_{\mu\nu} p_1 \cdot p_2\right)$$

$$\alpha^{\mu\nu}\beta_{\mu\nu} = 4^2\big(p_3\cdot p_2\, p_4\cdot p_1 + p_3\cdot p_1\, p_4\cdot p_2 - p_3\cdot p_4\, p_1\cdot p_2$$

$$+ p_4\cdot p_2\, p_1\cdot p_3 + p_4\cdot p_1\, p_3\cdot p_2 - p_4\cdot p_3\, p_1\cdot p_2$$

$$- p_1\cdot p_2\, p_3\cdot p_4 - p_1\cdot p_2\, p_3\cdot p_4 + 4\, p_1\cdot p_2\, p_3\cdot p_4\big)$$

$$= 4^2\left[2(p_3\cdot p_1)(p_4\cdot p_2) + 2(p_3\cdot p_2)(p_4\cdot p_1)\right]$$

$$= 2\cdot 4^2\left(p_3\cdot p_1\, p_4\cdot p_2 + p_3\cdot p_2\, p_4\cdot p_1\right) \quad \text{eq. } (7.126)$$

In CM frame

$$p_1 = (p, p\hat{z}) \qquad p_2 = (p, -p\hat{z})$$

$$p_3 = (p, \vec{p}) \qquad p_4 = (p, -\vec{p})$$

$$p_1\cdot p_3 = p^2 - p^2\cos\theta = p^2(1-\cos\theta)$$

$$p_4\cdot p_2 = p^2(1-\cos\theta)$$

$$p_3\cdot p_2 = p^2(1+\cos\theta)$$

$$p_4\cdot p_1 = p^2(1+\cos\theta)$$

$$k^2 = (p_1+p_2)^2 = 4p^2 = s \qquad \text{Mandelstam variable}$$

$$\langle m^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |m|^2$$

$$= \frac{1}{4} \frac{e^4}{(4p^2)^2} (2 \cdot 4^2) p^4 \left[(1-\cos\theta)^2 + (1+\cos\theta)^2 \right]$$

$$= \frac{e^4}{2} (2 + 2\cos^2\theta) = e^4 (1+\cos^2\theta)$$

$$d\sigma = \langle m^2 \rangle \frac{1}{4(P_1 \cdot P_2)} (2\pi)^4 \delta^4() \frac{d^3P_3}{2(2\pi)^3 P_3} \frac{d^3 P_4}{2(2\pi)^3 P_4}$$

$$\delta^3(\vec{P}_1 + \vec{P}_2 - \vec{P}_3 - \vec{P}_4) \Rightarrow \vec{P}_3 = -\vec{P}_4$$

$$\delta^0(2p - 2p') = \frac{1}{2}\delta^0(p - p')$$

$$\frac{d\sigma}{d\Omega} = \langle m^2 \rangle \frac{1}{4(2p^2)} \frac{1}{(2\pi)^2} \left(\frac{1}{2}\right)^2 \frac{1}{2}$$

$$= \frac{1}{4(4^3)} \frac{1}{\pi^2 p^2} e^4 (1+\cos^2\theta)$$

$$= \left(\frac{1}{4\pi}\right)^2 \frac{1}{4} \left(\frac{1}{4p^2}\right) (1+\cos^2\theta)$$

$$\boxed{\frac{e^2}{4\pi} = \alpha}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$

If we can ignore the Z