**Coherent State Worksheet**
**Intermediate Quantum 491**

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In class we showed that eigenstates of the harmonic oscillator creation operator (called coherent states)

$$\hat{a}|z\rangle = z|z\rangle$$
$$\langle z|\hat{a}^\dagger = z^*\langle z|$$

where $z$ is a complex number $z = (y_0 + iq_0)/\sqrt{2}$.

We found that

$$|z\rangle = e^{-|z|^2/2} e^{z\hat{a}^\dagger}|0\rangle$$

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1. Find $\langle\hat{y}\rangle$, $\Delta y$ and $\langle\hat{q}\rangle$, $\Delta q$ for coherent states and show that coherent states have the minimum uncertainty product $\Delta x \Delta p$ . Recall
$$\hat{y} = \hat{x}\sqrt{\frac{m\omega}{\hbar}}; \quad \hat{q} = \hat{p}\frac{1}{\sqrt{m\omega\hbar}}$$

2. Show that the probability distribution $|\langle n|z\rangle|^2$ is a Poisson with mean $|z|^2$.

3. Show that
$$|z\rangle = \exp\left(z\hat{a}^\dagger - z^*\hat{a}\right)|0\rangle$$
Hint: use the identity $e^{\hat{A}}e^{\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{A}+\hat{B}}$ for $[\hat{A},\hat{B}]$ = a number (not an operator).

4. Show that $\exp\left(z\hat{a}^\dagger - z^*\hat{a}\right)$ is equal to a translation in momentum space times a translation in position space (up to an irrelevant phase factor).

5. show that
$$\langle E\rangle = \hbar\omega\left(|z|^2 + \frac{1}{2}\right)$$
and that
$$\Delta E = \hbar\omega|z|$$
and that therefore $\Delta E/\langle E\rangle \to 0$ as $|z| \to \infty$

Physics 491
M. Gold

# Coherent State Worksheet

1/ $\Delta x \Delta p = \hbar \, \Delta y \Delta q$

let $z = \frac{1}{\sqrt{2}}(y_0 + i q_0)$ with $y_0, q_0$ real

recall $\hat{y} = \frac{1}{\sqrt{2}}(\hat{a}^+ + \hat{a})$ ; $\hat{q} = \frac{i}{\sqrt{2}}(\hat{a}^+ - \hat{a})$

$$\langle \hat{y} \rangle_z = \frac{1}{\sqrt{2}} \langle z | (\hat{a}^+ + \hat{a}) | z \rangle$$

recall $\hat{a}|z\rangle = z|z\rangle$ , $\langle z|\hat{a}^+ = \langle z| z^*$

then $\langle \hat{y} \rangle_z = \frac{1}{\sqrt{2}} \langle z | (z^* + z) | z \rangle = y_0$

similarly $\langle \hat{q} \rangle_z = \frac{i}{\sqrt{2}} \langle z | (\hat{a}^+ - \hat{a}) | z \rangle = q_0$

$$\langle \hat{y}^2 \rangle_z = \langle z | \tfrac{1}{2} (\hat{a}^+ + \hat{a})^2 | z \rangle$$

$$= \tfrac{1}{2} \langle z | \Big( \hat{a}^{+2} + \hat{a}^2 + \underbrace{\hat{a}^+\hat{a} + \hat{a}\hat{a}^+}_{2\hat{a}^+\hat{a}+1} \Big) | z \rangle$$

$$= \tfrac{1}{2}(z^* + z)^2 + \tfrac{1}{2} = y_0^2 + \tfrac{1}{2}$$

$$(\Delta y)^2 = \langle \hat{y}^2 \rangle - \langle \hat{y} \rangle^2 = \tfrac{1}{2}$$

similarly, $\langle \hat{q}^2 \rangle = q_0^2 + \frac{1}{2}$ , $(\Delta q)^2 = \frac{1}{2}$

thus $\Delta y \Delta q = \frac{1}{2}$ , $\Delta x \Delta p = \frac{\hbar}{2}$

2. $$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle = \frac{z^n}{\sqrt{n!}}e^{-|z|^2/2}$$

$$P_n = |\langle n|z\rangle|^2 = \frac{|z|^{2n}}{n!}e^{-|z|^2}$$

let $\mu = |z|^2$ then $P_N = \frac{\mu^N}{N!}e^{-\mu}$

which is a poisson with mean $\mu$:

$$\bar{n} = \sum n P_n = |z|^2$$

3./ $$|z\rangle = e^{-|z|^2/2} e^{z\hat{a}^+}|0\rangle$$

from $e^{\hat{A}}e^{\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{A}+\hat{B}}$

and $[z\hat{a}^+, z^*\hat{a}] = -|z|^2$

$$e^{-|z|^2/2}e^{z\hat{a}^+} = \exp(z\hat{a}^+ - z^*\hat{a})e^{+z^*\hat{a}}$$

now $e^{+z^*\hat{a}}|0\rangle = (1 + z^*\hat{a} + \dots)|0\rangle = |0\rangle$

so $|z\rangle = \exp(z\hat{a}^+ - z^*\hat{a})|0\rangle$

4./ rewrite $\exp\left(z\hat{a}^{+} - z^{*}\hat{a}\right)|0\rangle$

in terms of $\hat{y}, y_0, \hat{g}, g_0$

$$z\hat{a}^{+} = \frac{1}{2}(y_0 + i g_0)(\hat{y} - i\hat{g}) =$$

$$\frac{1}{2}\left(y_0 \hat{y} + i g_0 \hat{y} - i y_0 \hat{g} + g_0 \hat{g}\right)$$

$$\left(z\hat{a}^{+} - z^{*}\hat{a}\right) = i\left(g_0 \hat{y} - y_0 \hat{g}\right)$$

then since $\left[i g_0 \hat{y}, -y_0 \hat{g}\right] = g_0 y_0 [\hat{y}, \hat{g}] = i g_0 y_0$

$$\exp\left(z\hat{a}^{+} - z^{*}\hat{a}\right) = \exp\left(\frac{i g_0 y_0}{2}\right) \underbrace{\exp\left(i g_0 \hat{y}\right)}_{\text{translation by } g_0} \underbrace{\exp\left(-i y_0 \hat{g}\right)}_{\text{translation by } y_0}$$

5./ $$\langle E \rangle = \sum_{n=0}^{\infty} E_n \left|\langle n | z \rangle\right|^2$$

$$= \hbar\omega \sum \left(n + \frac{1}{2}\right) \frac{|z|^{2N}}{N!} e^{-|z|^2}$$

$$= \hbar\omega e^{-|z|^2} \left( \underbrace{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n-1)!}}_{|z|^2 e^{|z|^2}} + \frac{1}{2} \underbrace{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}}_{e^{|z|^2}} \right)$$

$$= \hbar\omega\left(|z|^2 + \frac{1}{2}\right)$$