Heisenberg's Microscope

M. Gold, physics 491

August 28, 2015

ıd the time t at which it is meas-
ı uncertainty principle states that
ıf the uncertainties in the knowl-
st Planck's constant h divided by
,[1] so that

(3.1)

(3.2)

(3.3)

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Mod. Phys. **37**, 537 (1965).
ıerits further attention; see M. M. Nieto,

Theory and the Description of Nature,"
; *Phys. Rev.* **48**, 696 (1935).

L. Schiff, Quantum Mechanics, 1968

elements are all necessary for the description of various aspects of the phenomena. From the point of view of the experimenter, the complementarity principle asserts that the physical apparatus available to him has such properties that more precise measurements than those indicated by the uncertainty principle cannot be made.

This is not to be regarded as a deficiency of the experimenter or of his techniques. It is rather a law of nature that, whenever an attempt is made to measure precisely one of the pair of canonical variables, the other is changed by an amount that cannot be too closely calculated without interfering with the primary attempt. This is fundamentally different from the classical situation, in which a measurement also disturbs the system that is under observation, but the amount of the disturbance can be calculated and taken into account. Thus the complementarity principle typifies the fundamental limitations on the classical concept that the behavior of atomic systems can be described independently of the means by which they are observed.

LIMITATIONS ON EXPERIMENT

In the atomic field, we must choose between various experimental arrangements, each designed to measure the two members of a pair of canonical variables with different degrees of precision that are compatible with the uncertainty relations. In particular, there are two extreme arrangements, each of which measures one member of the pair with great precision. According to classical theory, these extreme experimental arrangements complement each other; the results of both may be obtained at once and are necessary to supply a complete classical description of the system. In actuality, however, the extreme complementary experiments are mutually exclusive and cannot be performed together.

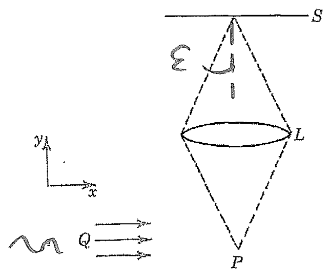
It is in this sense that the classical concept of causality disappears in the atomic field. There is causality insofar as the quantum laws that describe the behavior of atoms are perfectly definite; there is not, however, a causal relationship between successive configurations of an atomic system when we attempt to describe these configurations in classical terms.

4 DISCUSSION OF MEASUREMENT

In this section we consider three fairly typical measurement experiments from the point of view of the new quantum mechanics. The first two are designed to determine the position and momentum of a particle by optical methods; the third is the diffraction experiment in Sec. 2.

LOCALIZATION EXPERIMENT

We consider a particular example of the validity of the uncertainty principle, making use of a position-momentum determination that is typical



See ``Heisenberg's microscope"

Fig. 2 An experiment for the localization of a particle P by means of one of the scattered quanta Q, which is focused by the lens L to form an image on the screen S.

of a number of somewhat similar experiments that have been discussed in connection with measurements on particles and radiation fields.[1] We shall consider here the accuracy with which the x components of the position and momentum vectors of a material particle can be determined at the same time by observing the particle through a rather idealized microscope by means of scattered light.

The best resolving power of the lens L shown in Fig. 2 is known (either experimentally or from the theory of wave optics) to provide an accuracy

$$\Delta x \sim \frac{\lambda}{\sin \epsilon} \tag{4.1}$$

in a position determination, where λ is the wavelength of the radiation that enters the lens, and ϵ is the half angle subtended at the particle P by the lens. For simplicity, we consider the case in which only one of the light quanta Q is scattered onto the screen S. Because of the finite aperture of the lens, the precise direction in which the photon is scattered into the lens is not known. Then since Eq. (1.2) states that the momentum of the photon after it is scattered is h/λ,[2] the uncertainty in the x component of its momentum is approximately $(h/\lambda) \sin \epsilon$.

The x components of the momenta of the photon and the particle can be accurately known before the scattering takes place, since there is no need then to know the x components of their positions. Also, if our position measurement refers to the displacement of the particle with respect to the microscope, there is no reason why the total momentum of the system (particle, photon, and microscope) need be altered during the scattering. Then the uncertainty Δp_x in the x component of the momen-

[1] See, for example, W. Heisenberg, "The Physical Principles of the Quantum Theory," chaps. II, III (University of Chicago Press, Chicago, 1930); D. Bohm, "Quantum Theory," chap. 5 (Prentice-Hall, Englewood Cliffs, N.J., 1951).

[2] See footnote 2, page 3.

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tum of the particle after the scattering is equal to the corresponding uncertainty for the photon.

$$\Delta p_x \sim \frac{h}{\lambda} \sin \epsilon \qquad (4.2)$$

If we combine Eq. (4.1) with Eq. (4.2), we see that just after the scattering process

$$\Delta x \cdot \Delta p_x \sim h \qquad (4.3)$$

is the best that we can do for the particle. Thus a realistic accounting of the properties of the radiation gives a result in agreement with the uncertainty relation (3.1) for the particle.

This experiment may also be considered from the point of view of the complementarity principle. The complementary arrangements differ in the choice of wavelength of the observed radiation: Sufficiently small λ permits an accurate determination of the position of the particle just after the scattering process, and large λ of its momentum.

MOMENTUM DETERMINATION EXPERIMENT

The experiment just discussed assumes that the momentum of the particle is accurately known before the measurement takes place, and then it measures the position. It is found that the measurement not only gives a somewhat inaccurate position determination but also introduces an uncertainty into the momentum.

We now consider a different experiment in which the position is accurately known at the beginning and the momentum is measured. We shall see that the measurement not only gives a somewhat inaccurate momentum determination but also introduces an uncertainty into the position. We assume that the particle is an atom in an excited state, which will give off a photon that has the frequency ν_0 if the atom is at rest. Because of the doppler effect, motion of the atom toward the observer with speed v means that the observed frequency is given approximately by

$$\nu \approx \nu_0 \left(1 + \frac{v}{c}\right) \qquad (4.4)$$

so that

$$v \approx c \left(\frac{\nu}{\nu_0} - 1\right) \qquad (4.5)$$

Accurate measurement of the momentum mv by measurement of the frequency ν requires a relatively long time τ; the minimum error in the

Theoretical Critique: A P Lund and H M Wiseman,New Journal of Physics 12 (2010) was followed by an experiment, summarized for the non-expert Certainty_of_Uncertainty. Quoting from above reference,

“When first taking quantum mechanics courses, students learn about Heisenberg’s uncertainty principle, which is often presented as a statement about the intrinsic uncertainty that a quantum system must possess. Yet Heisenberg originally formulated his principle in terms of the observer effect: a relationship between the precision of a measurement and the disturbance it creates, as when a photon measures an electron’s position. Although the former version is rigorously proven, the latter is less general and as recently shown mathematically incorrect. In a paper in Physical Review Letters, Lee Rozema and colleagues at the University of Toronto, Canada, experimentally demonstrate that a measurement can in fact violate Heisenberg’s original precision-disturbance relationship.”

The experimental paper: PRL 109, 100404 (2012) attached at bottom

There is controversy, are-weak-values-quantum-dont-bet-it and paper Phys. Rev. Lett. 113, 120404 (2014) (by Christopher Ferrie and Joshua Combes, *Center for Quantum Information and Control, University of New Mexico, Albuquerque, New Mexico*) claiming that the Rozema result can be obtained classically and is therefore not quantum mechanics but classical statistics (a Monty Hall type paradox). Further theoretical analysis supporting Heisenberg: PRL 111, 160405 (2013)

“We have shown that despite recent claims to the contrary, Heisenberg-type inequalities can be proven that describe a trade-off between the precision of a position measurement and the necessary resulting disturbance of momentum and vice-versa.”

Professor Werner said: “Since I was a student I have been wondering what could be meant by an ’uncontrollable’ disturbance of momentum in Heisenberg’s Gedanken experiment. In our theorem this is now clear: not only does the momentum change, there is also no way to retrieve it from the post measurement state.”

Professor Lahti added: “It is impressive to witness how the intuitions of the great masters from the very early stage of the development of the then brand new theory turn out to be true.”

Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

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While there is a rigorously proven relationship about uncertainties intrinsic to any quantum system, often referred to as "Heisenberg's uncertainty principle," Heisenberg originally formulated his ideas in terms of a relationship between the precision of a *measurement* and the disturbance it must create. Although this latter relationship is not rigorously proven, it is commonly believed (and taught) as an aspect of the broader uncertainty principle. Here, we experimentally observe a violation of Heisenberg's "measurement-disturbance relationship", using weak measurements to characterize a quantum system before and after it interacts with a measurement apparatus. Our experiment implements a 2010 proposal of Lund and Wiseman to confirm a revised measurement-disturbance relationship derived by Ozawa in 2003. Its results have broad implications for the foundations of quantum mechanics and for practical issues in quantum measurement.

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The Heisenberg uncertainty principle is one of the cornerstones of quantum mechanics. In his original paper on the subject, Heisenberg wrote, "At the instant of time when the position is determined, that is, at the instant when the photon is scattered by the electron, the electron undergoes a discontinuous change in momentum. This change is the greater the smaller the wavelength of the light employed, i.e., the more exact the determination of the position" [1]. Here, Heisenberg was following Einstein's example and attempting to base a new physical theory only on observable quantities, that is, on the results of measurements. The modern version of the uncertainty principle proved in our textbooks today, however, deals not with the precision of a measurement and the disturbance it introduces, but with the *intrinsic* uncertainty any quantum state must possess, regardless of what measurement (if any) is performed [2–4]. These two readings of the uncertainty principle are typically taught side-by-side, although only the modern one is given rigorous proof. It has been shown that the original formulation is not only less general than the modern one—it is in fact mathematically incorrect [5]. Recently, Ozawa proved a revised, universally valid, relationship between precision and disturbance [6], which was indirectly validated in [7]. Here, using tools developed for linear-optical quantum computing to implement a proposal due to Lund and Wiseman [8], we provide the first direct experimental characterization of the precision and disturbance arising from a measurement, violating Heisenberg's original relationship.

In general, measuring one observable (such as position, q) will, according to quantum mechanics, induce a random disturbance in the complementary observable (in this case momentum, p). Heisenberg proposed, and it is widely believed, that the product of the measurement precision, $\epsilon(q)$, and the magnitude of the induced disturbance, $\eta(p)$, must satisfy $\epsilon(q)\eta(p) \approx h$, where h is Planck's constant. This idea was at the crux of the Bohr-Einstein debate [9], and the role of momentum disturbance in destroying interference has remained a subject of heated discussion [10–12]. Recently, the study of uncertainty relations in general has been a topic of growing interest, specifically in the setting of quantum information and quantum cryptography, where it is fundamental to the security of certain protocols [13,14]. The relationship commonly referred to as the Heisenberg uncertainty principle (HUP)—in fact proved later by Weyl [4], Kennard [3], and Robertson [2]—refers not to the precision and disturbance of a measurement, but to the uncertainties intrinsic in the quantum state. The latter can be quantified by the standard deviation $\Delta\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$, which is independent of any specific measurement. This relationship, generalized for arbitrary observables $\hat{A}$ and $\hat{B}$, reads

$$\Delta\hat{A}\Delta\hat{B} \geq \tfrac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|. \quad (1)$$

This form has been experimentally verified in many settings [15], and is uncontroversial. The corresponding generalization of Heisenberg's original measurement-disturbance relationship (MDR) would read

$$\epsilon(\hat{A})\eta(\hat{B}) \geq \tfrac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|. \quad (2)$$

This equation has been proven to be formally incorrect [5]. Recently, Ozawa proved that the correct form of the MDR in fact reads [6]

$$\epsilon(\hat{A})\eta(\hat{B}) + \epsilon(\hat{A})\Delta\hat{B} + \eta(\hat{B})\Delta\hat{A} \geq \tfrac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|. \quad (3)$$

Because of the two additional terms on the left-hand side, this inequality may be satisfied even when Heisenberg's MDR is violated.

Experimentally observing a violation of Heisenberg's original MDR requires measuring the disturbance and precision of a measurement apparatus (MA). While classically measuring the disturbance is straightforward—it simply requires knowing the value of an observable, $\hat{B}$, before and after the MA—quantum mechanically it seems impossible. Quantum mechanics dictates that any attempt to measure $\hat{B}$ before the MA must disturb $\hat{B}$ (unless the system is already in an eigenstate of $\hat{B}$); as we shall discuss later, it may also change the state in such a way that the right-hand side (RHS) of Heisenberg's inequality is modified as well. Because of these difficulties the disturbance, as described here, has been claimed to be experimentally inaccessible [16]. A recent experiment has indirectly tested Ozawa's new MDR [7], using a method also proposed by Ozawa [17]. Rather than directly characterizing the effects of an individual measurement, this work checked the consistency of Ozawa's theory by carrying out a set of measurements from which the disturbance could be inferred through tomographic means [18]; there has been some discussion on the arXiv site as to the validity of this approach [18–21]. In contrast, Lund and Wiseman showed that if the system is weakly measured [22,23] before the MA [Fig. 1(a)] the precision and disturbance can be directly observed in the resulting weak values [8]. Here we present an experimental realization of this proposal, directly measuring the precision of an MA and its resulting disturbance, and demonstrate a clear violation of Heisenberg's MDR.

To understand the definitions of the precision and disturbance we first describe our implementation of a variable-strength measurement. A variable-strength measurement can be realized as an interaction between the system and a probe followed by a strong measurement of the probe [24] [shaded area of Fig. 1(a)]. The system and probe become entangled through the interaction, disturbing the system, such that measuring the probe will yield information about

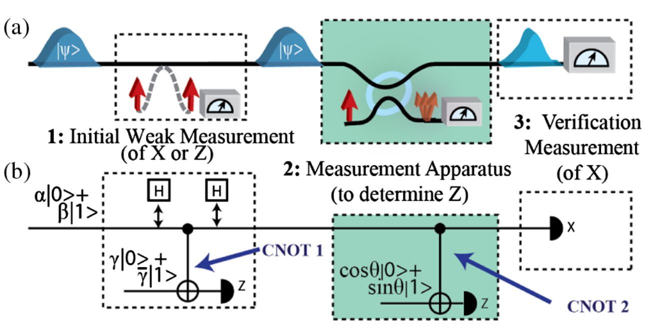

FIG. 1 (color online). The weak measurement proposal of Lund and Wiseman [8]. (a) A general method for measuring the precision and disturbance of any system. The system is weakly measured before the measurement apparatus and then strongly measured afterwards. (b) A quantum circuit which can be used to measure the precision and disturbance of $\hat{X}$ and $\hat{Z}$ for a qubit system.

the state of system. We define the disturbance as the root mean squared (rms) difference between the value of $\hat{B}$ on the system before and after the MA, while the precision is the RMS difference between the value of $\hat{A}$ on the system before the interaction and the value of $\hat{A}$ read out on the probe. Lund and Wiseman showed these rms differences can be directly obtained from a weak measurement on the system before the MA, post-selected on a projective measurement on either the probe or system afterwards [8]. Specifically, they showed that the precision and disturbance for discrete variables is simply related to the weak-valued probabilities of $\hat{A}$ and $\hat{B}$ changing, $P_{WV}(\delta\hat{A})$ and $P_{WV}(\delta\hat{B})$, via

$$\epsilon(\hat{A})^2 = \Sigma_{\delta\hat{A}}(\delta\hat{A})^2 P_{WV}(\delta\hat{A}), \tag{4}$$

$$\eta(\hat{B})^2 = \Sigma_{\delta\hat{B}}(\delta\hat{B})^2 P_{WV}(\delta\hat{B}). \tag{5}$$

By taking our system to be the polarization of a single photon we can demonstrate a violation of Heisenberg's precision limit by measuring one polarization component, $\hat{Z}$, and observing the resulting disturbance imparted to another, $\hat{X}$. Here, $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are the different polarization components of the photon; we use the convention that their eigenvalues are ± 1. For these observables, the bound [RHS of Eqs. (2) and (3)] of both Heisenberg and Ozawa's precision limits is $|\langle\hat{Y}\rangle|$. To facilitate the demonstration of a violation of Heisenberg's MDR, we make this bound as large as possible by preparing the system in the state $(|H\rangle + i|V\rangle)/\sqrt{2}$, so that $|\langle\hat{Y}\rangle| = 1$. In this state, the uncertainties are $\Delta\hat{X} = \Delta\hat{Z} = 1$, which satisfy Heisenberg's uncertainty principle [Eq. (1)], as they must. On the other hand, a measurement of $\Delta\hat{Z}$ can be made arbitrarily precise. Now, even if the Z precision, $\epsilon(\hat{Z})$, approaches zero the X disturbance, $\eta(\hat{X})$, to $\hat{X}$ can only be as large as $\sqrt{2}$, so that their product can fall below 1, violating Heisenberg's MDR. Note that attempting the same violation with the Heisenberg uncertainty principle, by setting $\Delta\hat{Z}$ to zero, requires that the system is prepared in either $|H\rangle$ or $|V\rangle$, in which case the bound, $|\langle\hat{Y}\rangle|$, must also go to zero, so that Eq. (1) is trivially satisfied.

We can measure $\hat{Z}$ of a single photon, by coupling it to a probe system with a quantum logic gate [25] [shaded region of Fig. 1(b)], implemented in additional path degrees of freedom of the photon [26]. We use this technique to implement both the weak measurement and the MA. Current linear-optical quantum gates are reliant on post-selection, which makes them prone to error [27]. We circumvent this problem, making use of ideas from the one-way model of quantum computing to implement the quantum circuit of Fig. 1(b) [28]. To enable successive CNOT gates between the system and the two probes we first make a "2-qubit line cluster" in the polarization of two photons.

Experimentally, we generate entangled 2-photon states of the form $\alpha|HH\rangle + \beta|VV\rangle$, using a spontaneous parametric down-conversion source in the "sandwich-configuration" [29]. Each crystal is 1 mm of BBO, cut for type-I phase matching. We can set α and β by setting the pump polarization with quarter- and half-wave plates. The pump beam is centered at 404 nm, with a power of 500 mW, generating down-converted photons at 808 nm. The pump is generated by frequency doubling a femtosecond Ti:sapphire laser, which is centered at 808 nm, using a 2 mm long crystal of BBO. The down-converted photons are coupled into single-mode fiber before being sent to the rest of the experiment. We observe approximately 15 000 entangled pairs a second, with 12% coupling efficiency, directly in the fiber. When coupling the light into multimode fiber after the interferometers, we measure about 1000 coincidence counts a second, spread among all the detector pairs. For each data point we acquire coincidence counts for 30 sec using a homebuilt coincidence counter based on an FPGA. We are able to make the desired entangled state with a fidelity of 95.9%, which we measure by performing quantum state tomography (QST) on the photons directly after the single-mode fiber [30].

A modified quantum circuit which implements Lund and Wiseman's proposal [8] and includes the line cluster creation is drawn in Fig. 2(a), with the corresponding optical implementation below in Fig. 2(b). A single logical polarization qubit, $\alpha|H\rangle + \beta|V\rangle$, is encoded in two physical polarization qubits, forming the line cluster $\alpha|H_1H_2\rangle + \beta|V_1V_2\rangle$. Using a line cluster allows the first photon's polarization to control a CNOT gate with an additional path degree of freedom, realized using a polarizing beam splitter (PBS), to implement the weak measurement. After

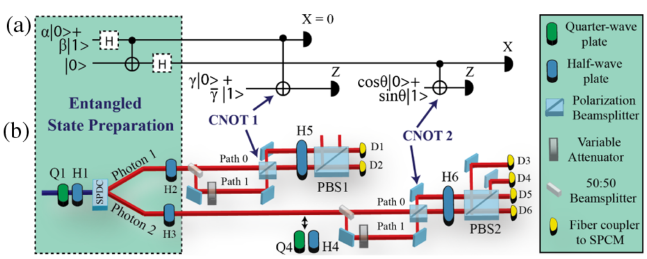

FIG. 2 (color online). (a) The logical quantum circuit that we implement. We use ideas from cluster state quantum computing, namely, single-qubit teleportation, to implement successive quantum gates. The first shaded area represents the creation of the entangled resource. After a 2-qubit cluster is created, the first qubit controls the controlled not gate used as our weak interaction. After this it is measured and its state is teleported to qubit 2. Qubit 2 then interacts with a second probe, which we use for our von Neumann measurement. (b) The optical setup we use to implement the quantum circuit in (a). We use two entangled photons generated from spontaneous parametric down-conversion as the first two qubits of the circuit. Path qubits are added to each photon with 50/50 beam splitters, and their state is initialized using variable attenuators.

this step the state is $\alpha|H_1H_2\rangle|A_1\rangle + \beta|V_1V_2\rangle|B_1\rangle$, where $|A_1\rangle$ and $|B_1\rangle$ denote two different states of the path degrees of freedom, $|A_1\rangle = \gamma|P_0\rangle + \bar{\gamma}|P_1\rangle$ and $|B_1\rangle = \bar{\gamma}|P_0\rangle + \gamma|P_1\rangle$. Now, measuring the first polarization in the $\hat{X}$ basis and finding $\hat{X} = +1$ teleports the state of the system to the polarization of the second photon, $\langle\frac{H_1+V_1}{\sqrt{2}}|(\alpha|H_1H_2\rangle|A_1\rangle + \beta|V_1V_2\rangle|B_1\rangle) = \alpha|H_2\rangle|A_1\rangle + \beta|V_2\rangle|B_1\rangle$. (If instead, the measurement result is $\hat{X} = -1$ the teleported state will be unitarily rotated to $\alpha|H_2\rangle|A_1\rangle - \beta|V_2\rangle|B_1\rangle$; in principle, one could correct this using feed-forward [31], but for simplicity we discard these events.) We characterize the teleportation by performing QST on the teleported single photon polarization. To do this we insert quarter- and half-wave plates, Q4 and H4, and remove the path qubit of photon 2. We find the teleported state has a fidelity of 93.4% with the expected state, mainly due to the reduced visibility of the interferometers. The polarization of the second photon is now free to be measured by the MA, which is implemented using a PBS and additional path degrees of freedom of photon 2, in the same way that photon 1 was weakly measured.

In order to clearly demonstrate a violation of Heisenberg's MDR we first experimentally characterize the bound of Eqs. (2) and (3). Lund and Wiseman discuss the limiting case of using perfectly weak measurements to characterize the system before the action of the MA [8]. However, in order to extract any information from this initial measurement, it cannot of course be infinitely weak. Although for our system, both the precision and the disturbance are independent of the weak measurement strength, the bound of Eqs. (2) and (3) is not. For instance, if we replaced the weak measurement of $\hat{Z}$ with a strong one, this would project the system onto eigenstates of $\hat{Z}$, all of which have $|\langle\hat{Y}\rangle| = 0$; the inequality would automatically be satisfied in this case. The weaker the measurement, the less $|\langle\hat{Y}\rangle|$ is reduced, and the stronger the inequality. We measured this experimentally, and Fig. 3 presents our data for $|\langle\hat{Y}\rangle|$ of the state just after the weak measurement, as a function of measurement strength, along with theory. It is important to note that these experimental difficulties can only lower the LHS of Eq. (2), and therefore cannot lead to a false violation.

To show a violation of Heisenberg's MDR we measure the precision and the disturbance of the MA. To measure the X disturbance we weakly measure $\hat{X}$ on the system before the MA post-selected on a strong measurement of $\hat{X}$ afterwards. Similarly, the Z precision of the MA is obtained by weakly measuring $\hat{Z}$ and then postselecting on a strong measurement of $\hat{Z}$ on the probe. From the results of these weak measurements the X disturbance and Z precision can be acquired. As an example, consider the X disturbance, $\eta(\hat{X})$, as defined in Eq. (5). We need to measure the quantities $P_{WV}(\delta\hat{X})$ for all $\delta\hat{X}$. Since we are dealing with the polarization of a single photon, $\delta\hat{X}$

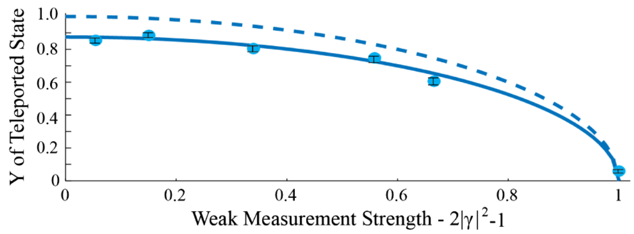

FIG. 3 (color online). A plot of the RHS of Eqs. (1) and (3) versus the strength of the weak probing measurement. The dashed line includes only the effect of the nonzero weak measurement strength. In addition to this effect, the solid line takes into account the imperfect teleportation.

can only equal 0 or ± 2. $P_{WV}(\delta\hat{X} = \pm 2)$ is the weak probability that the system initially had $\hat{X} = \mp 1$ and we found it in in $\hat{X} = \pm 1$. These probabilities can be expressed in terms of weak expectation values of $\hat{X}$, postselected on finding the system after the MA with $\hat{X}_f = \pm 1$, $\langle \hat{X} \rangle_{\hat{X}_f}$, as [8]: $P_{WV}(\delta\hat{X} = \pm 2) = \frac{1}{2}(1 \mp \langle \hat{X} \rangle_{\hat{X}_f = \pm 1}) P(\hat{X}_f = \pm 1)$. In our experiment, $P(\hat{X}_f = +1)$ corresponds to the probability of finding photon 2 diagonally polarized, given that the teleportation on the first photon's polarization succeeds, which is signalled by photon 1 being diagonally polarized. As shown in Fig. 2(b), both PBS's are set to measure in the diagonal basis, so this measurement amounts to counting two-photon events between the transmitted ports of PBS1 (detectors D1 or D2) and PBS 2 (detectors D5 or D6). The weak expectation value can be expressed in terms of the weak probe observable $\hat{Z}_p$, since $\hat{X}$ of the system couples to $\hat{Z}$ of the probe, as [25]:

$$\langle \hat{X} \rangle_{\hat{X}_f} = \frac{P(\hat{Z}_p = +1|\hat{X}_f) - P(\hat{Z}_p = -1|\hat{X}_f)}{2|\gamma|^2 - 1}. \quad (6)$$

Here, $2|\gamma|^2 - 1$ is the strength of the initial weak measurement, which we know and set through the state of the probe. The remaining quantities, $P(\hat{Z}_p = +1|\hat{X}_f = \pm 1)$ and $P(\hat{Z}_p = -1|\hat{X}_f = \pm 1)$, are directly measurable. For example, $P(\hat{Z}_p = +1|\hat{X}_f = \pm 1)$ is the probability of finding the first photon in path 1 given that the second photon was found vertically polarized in either path. It is measured by two-photon events between detector D1 (for the teleportation to succeed and for $\hat{Z}_p = +1$) and the transmitted port of PBS 2 (detectors D5 or D6), to postselect on $\hat{X}_f = +1$. A similar analysis can be done for the Z precision, but now rather than postselecting on the polarization of photon 2, $\hat{X}_f$, one has to postselect on the $\hat{Z}$ value of the MA probe, which is the path of the second photon.

The precision and disturbance were measured for several measurement apparatus strengths and are plotted in Fig. 4(a). The dashed lines are predictions for an ideal implementation of the quantum circuit in Fig. 2(a), while the solid lines, which fit our data well, take into account the

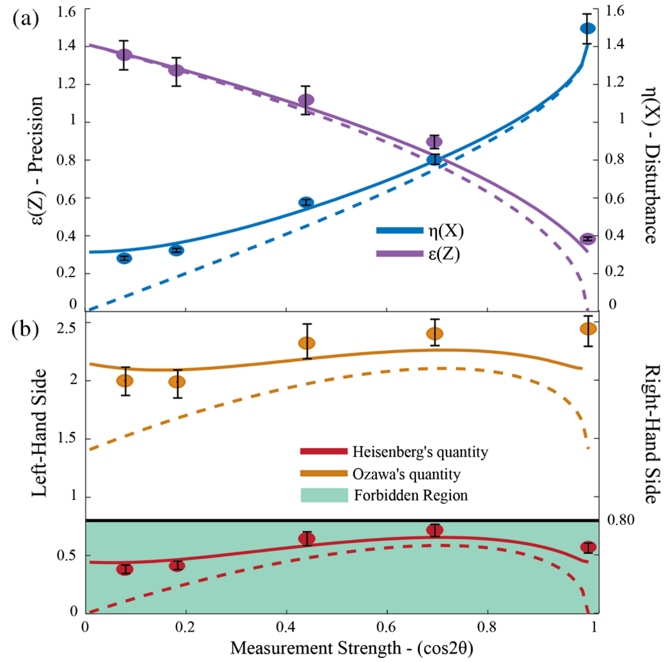

FIG. 4 (color online). Experimental results. (a) The precision of the measurement apparatus (MA) and disturbance it imparts to the system plotted against its strength. (b) A plot of the left-hand side of Heisenberg and Ozawa's relations versus the strength of the MA. For $\hat{X}$ and $\hat{Z}$ Heisenberg's quantity is $\epsilon(\hat{Z})\eta(\hat{X})$, and Ozawa's quantity is $\epsilon(\hat{Z})\eta(\hat{X}) + \epsilon(\hat{Z})\Delta\hat{X} + \eta(\hat{X})\Delta\hat{Z}$. The presumed bound on these quantities is given by the RHS of the relations, measured to be $|\langle \hat{Y} \rangle| = 0.80 \pm 0.02$. Heisenberg's MDR is clearly violated, with his quantity falling below the bound, while Ozawa's MDR remains valid for all experimentally accessible parameters.

imperfect entangled state preparation. The imperfect state preparation leads to errors in the single-qubit teleportation, increasing the rms difference between the measurements on the weak probe before the MA and the final verification measurements, on the system and probe, after the MA. Again, these errors can only increase disturbance and precision, and thus the LHS of Eq. (2), and cannot lead to a false violation.

From the measured precision and disturbance the LHS of Heisenberg and Ozawa's precision limits can be constructed. We set the strength of the initial weak measurement such that the RHS of Eq. (2) is large enough that Heisenberg's MDR violated for all settings of the MA. We measure $|\langle \hat{Y} \rangle| = 0.80 \pm 0.02$, which gives the forbidden region in Fig. 4(b). Heisenberg's quantity, which can be reconstructed simply from the measurements of the precision and the disturbance, is plotted in red. Ozawa's quantity, for which additional measurements of $\Delta\hat{X}$ and $\Delta\hat{Z}$ were made on the state, using quarter- and half-wave plates Q4 and H4, after the weak measurement, is plotted in orange. The error bars are due to Poissonian counting statistics. As seen in Fig. 4(b), Ozawa's MDR remains valid for all the experimentally tested parameters, while we find that the simple product of the precision and the disturbance—Heisenberg's MDR—always falls below the experimentally measured bound.

In conclusion, using weak measurements to experimentally characterize a system before and after it interacts with a measurement apparatus, we have directly measured its precision and the disturbance. This has allowed us to measure a violation of Heisenberg's hypothesized MDR. Our work conclusively shows that, although correct for uncertainties in states, the form of Heisenberg's precision limit is incorrect if naively applied to measurement. Our work highlights an important fundamental difference between uncertainties in states and the limitations of measurement in quantum mechanics.

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Proof of Heisenberg's Error-Disturbance Relation

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While the slogan ''no measurement without disturbance'' has established itself under the name of the Heisenberg effect in the consciousness of the scientifically interested public, a precise statement of this fundamental feature of the quantum world has remained elusive, and serious attempts at rigorous formulations of it as a consequence of quantum theory have led to seemingly conflicting preliminary results. Here we show that despite recent claims to the contrary [L. Rozema *et al*, Phys. Rev. Lett. **109**, 100404 (2012)], Heisenberg-type inequalities can be proven that describe a tradeoff between the precision of a position measurement and the necessary resulting disturbance of momentum (and vice versa). More generally, these inequalities are instances of an uncertainty relation for the imprecisions of *any* joint measurement of position and momentum. Measures of error and disturbance are here defined as figures of merit characteristic of measuring devices. As such they are state independent, each giving worst-case estimates across all states, in contrast to previous work that is concerned with the relationship between error and disturbance in an individual state.

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In spite of their important role since the very beginning of quantum mechanics, uncertainty relations have recently become the subject of active scientific debates. On one hand, entropic versions of the information-disturbance tradeoff [1] have become an important tool in security proofs [2] for continuous variable cryptography. On the other hand, there were widely publicized [3] claims of a refutation [4–6] of the error-disturbance uncertainty relations heuristically claimed by Heisenberg [7]. A review of the literature on uncertainty relations is given in [8].

Heisenberg's 1927 paper [7] introducing the uncertainty relations is one of the key contributions to early quantum mechanics. It is part of virtually every quantum mechanics course, almost always in the version forwarded by Kennard [9], Weyl [10], and Robertson [11]. What is often overlooked, however, is that this popular version is only one way of making the idea of uncertainty precise. The original paper begins with a famous discussion of the resolution of microscopes, in which the accuracy (resolution) of an approximate position measurement is related to the disturbance of the particle's momentum.

This situation is in no way covered by the standard relations, since in an experiment concerning the Kennard-Weyl-Robertson inequality no particle meets with both a position and a momentum measurement. Heisenberg's semiclassical discussion has no immediate translation into the modern quantum formalism, particularly since the momentum disturbance *prima facie* involves the comparison of two (generally) noncommuting quantities, the momentum before and after the measurement. Such a translation does require some careful conceptual work, and one can arrive at different results. This is shown by the example of Ozawa [4], who defines a relation he claims to be a rigorous version of Heisenberg's ideas, and shows that it fails to hold in general. A suggested modification of the false relation has recently been verified experimentally [5,6]. This has been widely publicized as a refutation of Heisenberg's ideas, in apparent contradiction to our main result. However, there is no contradiction, and the disagreement only shows that there is a grain of rigorously explicable truth in Heisenberg, provided one looks in the right place for it. While Ozawa aims to describe the interplay between error and disturbance for an individual state, our approach gives a state-independent characterization of the overall performance of measuring devices. In [12] we show that Ozawa's notions, though mathematically well defined, have only limited validity as measures of error and disturbance [13].

We will describe and prove an inequality of the classic form

$$(\Delta Q)(\Delta P) \geq \frac{\hbar}{2}, \qquad (1)$$

in which the quantities ΔQ and ΔP are *not* given by the variances of the position and momentum distributions in the same state, as in the textbook inequality. Instead, following closely the suggestion of Heisenberg, they are explicitly defined figures of merit for a microscopelike measurement scenario: the accuracy ΔQ of a position measurement and the momentum disturbance ΔP incurred by it. Moreover, the inequality is sharp, and we will describe explicitly the cases of equality. We believe that the definitions and results are simple enough to use in a basic quantum mechanics course, although the full proof uses some tools beyond such a course.

The main progress over earlier work [14] is a simpler definition of the Δ quantities, using the idea of calibration [16]. This definition does not require the Monge transportation metric, which led in [14] to quantities akin to absolute deviations rather than root mean square deviations, and hence to a constant different from $\hbar/2$ in (1). A changed constant (even if optimal for the particular definitions of Δ) puts an undue burden on the memory of undergraduates. Using variances also for calibration solves this problem. The basic ideas of the proof in [14] can be taken over.

To keep matters simple, we stick to the classic situation of two canonically conjugate variables of a single quantum degree of freedom. For the sake of comparison, let us recall the scenario of the Kennard-Weyl-Robertson inequality, which we call preparation uncertainty (see Fig. 1). The spreads $\Delta_\rho(A) = [\mathrm{tr}\rho A^2 - (\mathrm{tr}\rho A)^2]^{1/2}$ of position Q and momentum P are determined in separate experiments on the same source, given by a density operator ρ. The uncertainty relation $\Delta_\rho(Q)\Delta_\rho(P) \geq \hbar/2$ is a quantitative version of the observation that there are no dispersion-free quantum states [17], as applied to a canonical pair of observables. It is *not* to be found in Heisenberg's paper [7], except in a rough discussion of postmeasurement states, which he assumes to be Gaussian with a spread related to the accuracy of a position measurement.

In contrast, Fig. 2 shows the scenario discussed by Heisenberg. The middle row shows an approximate position measurement Q' followed by a momentum measurement. How should we define the momentum disturbance and position error in this setup? The error of the approximate position measurement Q' clearly refers to the comparison with an ideal measurement Q as shown in the first row. For the momentum disturbance we can say the same: We have remarked that the momenta before and after the microscope interaction do not commute, so the difference makes no sense in the individual case. However, we can compare the *distributions* of the momenta measured after the position measurement (we call this effective measurement P') with the *distribution* an ideal momentum measurement P would have given on the same input state. Come to think of it, this is precisely how we detect disturbance in other typical quantum settings. Consider, for example, the double slit experiment. It is well known that illuminating the slits enough to detect the passage of a particle through one or the other hole makes the interference fringes go away. Clearly, the light used for observation disturbs the particles, and the evidence for this is once again the change of the distribution on the screen. Note that this way of looking at error and disturbance restores the symmetry between the position and momentum aspects of this scenario. The uncertainty relations we will prove therefore apply just as well to the position disturbance caused by an approximate momentum measurement and, more generally, to any measurement scheme M, which produces in every run a value p and a value q (see the dashed outline in Fig. 2). This generalization also covers any successive measurement scenario, in which one tries to correct for some of the momentum disturbance, perhaps using the detailed knowledge of how the position measuring device works. In principle, this could allow a reduction of uncertainties. However, the inequality holds without change, which gives a precise meaning and a proof to Heisenberg's phrase "uncontrollable momentum disturbance," which he himself uses without further justification.

Let us now discuss the definition of $\Delta(Q, Q')$ in more detail (the momentum case will be completely analogous). We think of this "microscope resolution" as a figure of merit for the device, a promise which might be advertised by the manufacturer, and which could be verified by a testing lab. $\Delta(Q, Q') = 0$ will mean that the "approximate" device Q' is completely equivalent to the ideal Q; i.e., for every input state ρ the output distributions will be

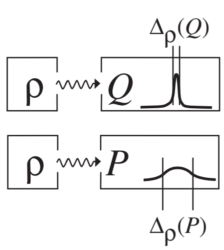

FIG. 1. Scenario of preparation uncertainty. Δ_ρ is the root of the variance of the distribution obtained for the indicated observable in the state ρ. In this pair of experiments no particle is subject to both a position and a momentum measurement.

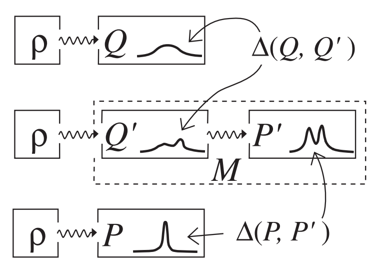

FIG. 2. Scenario of measurement uncertainty for successive measurements, as discussed by Heisenberg (middle row). An approximate position measurement Q' is followed by an ideal momentum measurement, effectively given a measurement P' on the initial state. The accuracy $\Delta(Q, Q')$ quantifies the difference between the output distributions of Q' and an ideal position measurement Q (first row). Similarly, the momentum disturbance $\Delta(P, P')$ quantifies the difference between the distributions obtained by P' and by an ideal momentum measurement P (last row). The definitions for these Δ quantities (see text) can be applied, more generally, to an arbitrary joint measurement M (dashed box). This can be any device producing, in every shot, a q value and a p value. Q' and P' are then defined as the marginals of M, obtained by ignoring the other output.

the same. Similarly, a small value might indicate that the difference in the distributions will be small for every input state. This requires a definition for the distance of two general probability distributions, which we will give below (see the section labeled "Uncertainty metrics"). However, we can also take a simpler approach, which avoids verifying a statement for *all* input states. Instead, the testing lab might concentrate on those states, which at least classically would seem to be the most demanding ones, namely, states for which Q has a known and sharp value. We call this process "calibration." Still, this requires testing of many states but no longer on very mixed states, or states which contain coherent superpositions of widely separated wave functions.

An advantage of the calibrated error is that we no longer need a quantitative evaluation of the distance between arbitrary probability distributions, but just between an arbitrary distribution and a known sharp value ξ. For this we naturally take the root mean square deviation from ξ,

$$D(\rho, Q'; \xi) = \langle (q' - \xi)^2 \rangle^{1/2}_{\rho, Q'}, \tag{2}$$

where the angle brackets denote the expectation of the indicated function of the output q', in the distribution obtained on the preparation ρ with the device Q'. This statement allows for Q' to be a general positive operator valued measurement. For projection valued observables like Q we could simplify this to $D(\rho, Q; \xi)^2 = \mathrm{tr}[\rho(Q - \xi \mathbb{1})^2]$. The latter quantity is to be small, say $\leq \varepsilon$, for the input states ρ used for calibration. Hence, we set $\Delta_c(Q, Q')$ to be

$$\lim_{\varepsilon \to 0} \sup\{D(\rho, Q'; \xi) | \rho, \xi; D(\rho, Q; \xi) \leq \varepsilon\}. \tag{3}$$

Here, the set is nonempty, since for any ξ and $\epsilon > 0$ there is a ρ such that $\mathrm{tr}(\rho Q) = \xi$ and $D(\rho, Q; \xi) < \epsilon$; moreover, the limit exists, because with decreasing ε the supremum is over fewer and fewer states, so the function is nonincreasing. In the case of a bad approximation, the supremum can be infinite, in which case we put $\Delta_c(Q, Q') = \infty$.

With this definition, and the corresponding one for P, we can state our main result. We just assume that the Q' and P' are the marginal observables of some joint measurement device M whose calibration errors are both finite. As discussed above, this also covers the case of a sequential measurement (Fig. 2). Then

$$\Delta_c(Q, Q')\Delta_c(P, P') \geq \frac{\hbar}{2}. \tag{4}$$

This inequality is sharp, and equality holds for an M for which the joint distribution of (q, p) outputs is the so-called Husimi distribution [18] of the input state, which can be obtained by a Gaussian smearing of the Wigner function. In the extreme case of one of the marginals being error free, the error for the other marginal is necessarily infinite.

Proof.—The proof has two parts: The first is elementary and concerns the special case that M is a covariant phase space observable. These observables [18–21] can be described explicitly, including a very simple form of their marginals Q' and P', by which (4) can be reduced to the preparation uncertainty. The second, more technical part of the proof reduces the general case to the covariant case by an averaging method, and is taken from [14]. We only sketch it [22].

By a covariant measurement we mean one which has a natural symmetry property for both position and momentum translations. That is, if we apply it to an input state shifted in position by δq and in momentum by δp, the output distribution will be the same as before, transformed by $(q, p) \mapsto (q + \delta q, p + \delta p)$. These symmetries are implemented by the Weyl operators (also known as Glauber translations) $W(q, p) = \exp[(iqP - ipQ)/\hbar]$. Then the whole observable can be reconstructed from its density at the origin, which must be [20,21] a positive operator σ of trace 1, i.e., a density operator as for a quantum state. The probability for outcomes in a set $S \subseteq \mathbb{R}^2$ is then given by the positive operator

$$M(S) = \int_S \frac{dq dp}{2\pi\hbar} W(q, p)^* \sigma W(q, p). \tag{5}$$

A remarkable property of these joint measurements of position and momentum is that their marginals take a particularly simple form: The probability density of the outputs q' obtained on a state ρ is a convolution of the position distributions of ρ and σ. That is, we can model the output distribution by taking q distributed like the outputs of an ideal measurement Q on ρ, and adding a noise term q'', which is independent of q and distributed according to the position distribution of σ. The same description applies to the marginal P'.

Therefore, for a covariant measurement we can immediately identify $\Delta_c(Q, Q')$ without further computation: The density σ is a fixed characteristic property of the measurement. Therefore, as the position distribution of ρ becomes sharply concentrated around some ξ, the outputs converge in distribution to $q' = \xi + q''$, so

$$\Delta_c(Q, Q') = D(\sigma, Q; 0), \tag{6}$$

which is the "size" (the root mean square deviation) of the "noise." For example, if σ has sharp position distribution at some value a, this is equal to $|a|$, since the outputs will be off by a shift a (i.e., $q' \approx q + a$). Hence, one will choose σ with zero mean. The uncertainty product then becomes $\Delta_c(Q, Q')\Delta_c(P, P') = \Delta_\sigma(Q)\Delta_\sigma(P)$, which is $\geq \hbar/2$ by the preparation uncertainty relation applied to σ. This proves Eq. (4) for the case of covariant measurements, and at the same time provides examples of minimum uncertainty measurements: all we have to do is to choose σ as a centered minimum uncertainty state, i.e., as $\sigma = |\Psi\rangle\langle\Psi|$ with Ψ a real valued centered Gaussian wave function. The phase space distribution associated with an input state ρ by this measurement M is then the Husimi distribution [18].

The more technical part of the proof of Eq. (4) is to show that for any measurement M there is a covariant one, say $\bar{M}$, with at most the same Δ's. Basically, $\bar{M}$ is obtained from M by averaging, the technical problem being that the parameter range of (q, p) over which one has to "average" is infinite (see [14]). Let us introduce $\mathcal{M}_\varepsilon(\Delta Q, \Delta P)$ as the set of measurements M such that, for $A = Q, P$, $D(\rho, A'; \xi) \le \Delta A$ whenever $D(\rho, A; \xi) \le \varepsilon$ for given ΔA and ε. This is a convex set, and compact in a suitable weak topology. We can write the covariance condition as a fixed point equation for some transformations on the set of all observables, namely, a unitary transformation by a Weyl operator combined with a shift in the argument. These transformations commute, and leave $\mathcal{M}_\varepsilon(\Delta Q, \Delta P)$ invariant. Therefore, by the Markov-Kakutani fixed point theorem this set, if nonempty, must also contain a covariant element, which by construction has at most the same uncertainties. This concludes our sketch of the proof of Eq. (4).

Uncertainty metrics.—The calibration criterion only involves highly concentrated states so that, in principle, on general input states the optimal joint measurement might produce output distributions quite different from the ideal ones. One can easily give examples of a projection valued observable A and an "approximation" A' for which the calibrated distance is a rather optimistic estimate. That is, if we denote by $\Delta(Q, Q')$ a figure of merit based on comparison of *all* states, we might have $\Delta(Q, Q') \gg \Delta_c(Q, Q')$. Note first that in the covariant case this cannot happen: The statement that Q' can be simulated by adding fixed independent noise to Q is valid for arbitrary input states, and any reasonable definition of $\Delta(Q, Q')$ should give the size of the noise. However, in the general case we would need a definition which is independent of that special form. Here we will introduce such a quantity and show that an uncertainty relation holds for it.

The idea is to define a metric D on probability distributions which extends (2) in the sense that $D(\rho, Q'; \xi)$ becomes the metric distance between the output distribution of Q' and a point measure at ξ. Then we set

$$\Delta(Q, Q') = \sup_\rho D(\rho, Q; \rho, Q'), \tag{7}$$

where the expression on the right-hand side is the metric distance of the two output distributions. Since Δ_c takes the supremum over the smaller set of highly concentrated states, we have $\Delta(Q, Q') \ge \Delta_c(Q, Q')$. The metric D on probability distributions is basically fixed by our requirements as what is technically known as the Wasserstein-2 distance, which is a variant of the Monge-Kantorovich transport or "earth mover's" distance (see [24] for a study of such metrics). The problem addressed by Monge was the cost of transforming a hill (earth distribution μ) into some fortifications (earth distribution η), when the workers had to be paid by the bucket and the distance covered. A transport plan, also known as a *coupling* between the measures μ and η, would be a measure γ on $\mathbb{R} \times \mathbb{R}$ describing how much earth was to be moved from x to y. This entails that the marginals of γ must be μ and η. The cost in the Monge problem is $\int \gamma(dxdy)|x - y|$, which is then minimized by choosing an optimal γ. In the Wasserstein-2 distance the cost function is chosen to be quadratic in the distance and an overall root is taken to bring the units back to a length

$$D(\mu; \eta) = \inf_\gamma \left(\int \gamma(dxdy)|x - y|^2 \right)^{1/2}, \tag{8}$$

where the infimum is over all couplings γ. Consider now the case that η arises from μ by adding independent noise with distribution ν, which amounts to the convolution $\eta = \mu * \nu$. This immediately suggests a transport plan, namely, shifting each individual element of the μ distribution by the amount suggested by the noise [formally, $\gamma(dxdy) = \mu(dx)\nu[d(y - x)]$]. This may not be optimal, but gives the estimate $D(\mu; \mu * \nu) \le D(\nu; 0)$, the size of the noise, where once again the second argument stands for the point measure at zero. This says that the largest distance is attained for a point measure μ, and therefore

$$\Delta(Q, Q') = \Delta_c(Q, Q') \tag{9}$$

whenever Q' is the marginal of a covariant measurement. To summarize this section. if we define the deviation between Q and Q' by a worst-case figure of merit over *all* states, the uncertainty relation once again holds. Moreover, the two notions coincide on all covariant measurements, and in particular for the cases of equality.

Conclusion and outlook.—With the inequality (4) we have provided a general, quantitative quantum version of Heisenberg's original semiclassical uncertainty discussion. This is a remarkable vindication of Heisenberg's intuitions, far beyond the usual view, which takes the quantitative content of the paper to be summarized entirely by the preparation inequality, and sees the discussion of the microscope as no more than a heuristic order of magnitude argument.

Our conceptual framework applies to any pair of observables which are not jointly measurable. However, evaluating the respective uncertainty bounds, which will typically not be expressed in terms of the product of uncertainties, is another matter requiring further studies.

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