Physics 491

HW #10 Solutions

6.1(a) Use $[\hat{X}, \hat{P}_x]$ to exchange $\hat{X}\hat{P}_x$ as

$$\hat{X}^N \hat{P}_x = \hat{X}^{N-1}(\hat{X}\hat{P}_x) = i\hbar \hat{X}^{N-1} + \hat{X}^{N-1}\hat{P}_x \hat{X} \quad (1^{st} \text{ exchange})$$

$$\hat{X}^{N-1}\hat{P}_x = i\hbar \hat{X}^{N-2} + \hat{X}^{N-2}\hat{P}_x \hat{X}$$

so $\hat{X}^{N-1}\hat{P}_x \hat{X} = i\hbar \hat{X}^{N-1} + \hat{X}^{N-2}\hat{P}_x \hat{X}^2$

and $\hat{X}^N \hat{P}_x = 2i\hbar \hat{X}^{N-1} + \hat{X}^{N-2}\hat{P}_x \hat{X}^2$ (2 exchanges)

thus, by induction, after N exchanges of $\hat{X}\hat{P}_x$,

$$\hat{X}^N \hat{P}_x = N i\hbar \hat{X}^{N-1} + \hat{P}_x \hat{X}^N$$

$$[\hat{X}^N, \hat{P}_x] = N i\hbar \hat{X}^{N-1}$$

(b) Write $F(\hat{X}) = \sum_{N=0}^{\infty} \frac{X^N}{n!} \frac{d^N F}{dX^N}\Big|_{\hat{X}=X^0}$

$\hat{X}$ commutes with $F(X_0)$, $d^N F/dX^N\big|_{X_0}$ so that

$$[F(\hat{X}), \hat{P}_x] = \sum_{N=1}^{\infty} \frac{1}{N!}[\hat{X}^N, \hat{P}_x] \frac{d^N F}{dX^N}\Big|_{X_0}$$

$$= \sum_{N=1}^{\infty} \frac{i\hbar}{(N-1)!} \hat{X}^{N-1} \frac{d^N F}{dX^N}\Big|_{X_0}$$

$$= i\hbar \sum_{m=0}^{\infty} \frac{\hat{X}^m}{m!} \frac{d^{m+1} F}{dX^{m+1}}\Big|_{X_0} = i\hbar \frac{\partial F}{\partial X}(\hat{X})$$

(c) $\frac{d}{dt}\langle \hat{P}_x \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{P}_x] \rangle$

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + V(\hat{x})$$

$$[\hat{H}, \hat{P}_x] = [V(\hat{x}), \hat{P}_x] = i\hbar \frac{\partial V(\hat{x})}{\partial \hat{x}}$$

thus $\boxed{\frac{d\langle \hat{P}_x \rangle}{dt} = -\left\langle \frac{\partial V}{\partial x} \right\rangle}$

6.2 we know $\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle$

where $\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$

insert the identity operator $\hat{1} = \int |x\rangle\langle x| dx$

$$\langle p|\hat{x}|\psi\rangle = \int dx \langle p|x\rangle\langle x|\hat{x}|\psi\rangle$$

$$= \int dx \; x\langle p|x\rangle\langle x|\psi\rangle$$

but $x\langle p|x\rangle = i\hbar\frac{\partial}{\partial p}\frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar} = i\hbar\frac{\partial}{\partial p}\langle p|x\rangle$

thus $\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\int dx \langle p|x\rangle\langle x|\psi\rangle$

$$= i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

Use the identity as $1 = \int |p\rangle\langle p| dp$

$$\langle \phi|\hat{x}|\psi\rangle = \int dp \langle\phi|p\rangle\langle p|\hat{x}|\psi\rangle$$

$$= \int dp \langle\phi|p\rangle\left(i\hbar\frac{\partial}{\partial p}\right)\langle p|\psi\rangle$$

$$= \int dp \langle p|\phi\rangle^*\left(i\hbar\frac{\partial}{\partial p}\right)\langle p|\psi\rangle$$

this suggest $\hat{x} \xrightarrow[\text{momentum space}]{} i\hbar\frac{\partial}{\partial p}$

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or, the matrix representation of the $\hat{x}$ operator

or
$$\langle p' | \hat{x} | p \rangle = i\hbar \frac{\partial}{\partial p} \delta(p - p')$$

So for example,
$$\langle p | \hat{x} | \psi \rangle = \int dp' \, \langle p | \hat{x} | p' \rangle \langle p' | \psi \rangle$$
$$= i\hbar \frac{\partial}{\partial p} \int dp' \, \delta(p - p') \langle p' | \psi \rangle$$
$$= i\hbar \frac{\partial}{\partial p} \langle p | \psi \rangle$$

and
$$\langle \phi | \hat{x} | \psi \rangle = \int dp \int dp' \, \langle \phi | p \rangle \langle p | \hat{x} | p' \rangle \langle p' | \psi \rangle$$
$$= \int dp \int dp' \, \langle \phi | p \rangle \, i\hbar \frac{\partial}{\partial p} \delta(p - p') \langle p' | \psi \rangle$$
$$= \int dp \, \langle \phi | p \rangle \left(i\hbar \frac{\partial}{\partial p} \right) \left(\langle p | \psi \rangle \right)$$

6.3. We can show this in general, not simply for infinitesimal $\hat{T}$, using the commutator derived in class:

$$\left[\hat{x}, \hat{T}(a)\right] = a\hat{T}(a)$$

$$\langle\psi'|\hat{x}|\psi'\rangle = \langle\psi|T^{+}\hat{x}T|\psi\rangle$$

$$= \langle\psi|\hat{T}^{+}\left(\hat{T}\hat{x} + a\hat{T}\right)|\psi\rangle$$

since $\hat{T}^{+}\hat{T} = \hat{1}$, we have

$$\langle\hat{x}\rangle' = \langle\hat{x}\rangle + a$$

Since $[\hat{p}, \hat{T}] = 0$, we immediately have

$$\langle\hat{p}\rangle' = \langle\psi|\hat{T}^{+}\hat{p}\hat{T}|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle = \langle\hat{p}\rangle$$

6.8

6.7 (a) to show that $\hat{x}$ is Hermitian, insert $\hat{1} = \int dx\, |x\rangle\langle x|$

$$\langle \phi | \hat{x} | \psi \rangle = \int dx\, \langle \phi | \hat{x} | x \rangle \langle x | \psi \rangle$$

$$= \int dx\, \langle \phi | x \rangle\, x\, \langle x | \psi \rangle \quad \text{using } \hat{x}|x\rangle = x|x\rangle$$

taking the complex conjugate (x is a real number)

$$\langle \phi | \hat{x} | \psi \rangle^* = \int dx\, \langle \phi | x \rangle^*\, x\, \langle x | \psi \rangle^*$$

$$= \int dx\, \langle \psi | x \rangle\, x\, \langle x | \phi \rangle = \langle \psi | \hat{x} | \phi \rangle$$

(b) $\hat{p}$ is the generator of translations and must be Hermitian. We can prove it in the $\hat{x}$-basis:

$$\langle \phi | \hat{p} | \psi \rangle = \int dx\, \langle \phi | x \rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | \psi \rangle$$

$$\langle \phi | \hat{p} | \psi \rangle^* = \int dx \left(-\frac{\hbar}{i} \frac{\partial}{\partial x} \langle \psi | x \rangle \right) \left(\langle x | \phi \rangle \right)$$

$$= \int dx\, \langle \psi | x \rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | \phi \rangle \quad \text{integrating by parts}$$

$$= \langle \psi | \hat{p} | \phi \rangle$$

Now using $(AB)^\dagger = B^\dagger A^\dagger$ (think matrices)

$$[\hat{x}, \hat{p}] = i\hbar$$

take the conjugate of this equation to get

$$(\hat{x}\hat{p})^\dagger - (\hat{p}\hat{x})^\dagger = -i\hbar$$

$$\hat{p}\,\hat{x}^\dagger - \hat{x}^\dagger p = -i\hbar \qquad (\hat{p}^\dagger = \hat{p})$$

$$[\hat{x}^\dagger, \hat{p}] = i\hbar \Rightarrow \hat{x}^\dagger = \hat{x}$$

not assigned Fall 2009

6.14 not assigned Fall 2015

~~6.11~~ For the ground state of a particle in a box $(0<x<L)$:

$$\psi_1(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) & 0<x<L \\ 0 & x>L \text{ or } x<0 \end{cases}$$

$$\langle x\rangle = \frac{2}{L}\int_0^L dx\, x\sin^2\left(\frac{\pi x}{L}\right) = \left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)^2 \underbrace{\int_0^\pi dy\, y\sin^2 y}_{\pi^2/4}$$

$$= \left(\frac{2}{L}\right)\left(\frac{L^2}{\pi^2}\right)\frac{\pi^2}{4} = \frac{L}{2}$$

$$\langle x^2\rangle = \frac{2}{L}\int_0^L dx\, x^2\sin^2\left(\frac{\pi x}{L}\right) = \left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)^3\int_0^\pi dy\, y^2\sin^2 y$$

the integral is $I = \int_0^\pi dy\, y^2\sin^2 y = -\int_0^\pi dy\left(\frac{y^3}{3}\right)2\sin y\cos y$

integrate by parts

$$= -\int_0^\pi dy\,\frac{y^3}{3}\sin(2y) = -\frac{1}{3}\left(\frac{1}{2}\right)^4\int_0^{2\pi} dx\, x^3\sin x$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^4(x^3 - 6x)\cos x\Big|_0^{2\pi} = \left(\frac{1}{6}\pi^3 - \frac{1}{4}\pi\right)$$

table of integrals

so $\langle x^2 \rangle = \frac{2L^2}{\pi^3}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right) = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right)$

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = L^2\left[\frac{1}{3} - \frac{1}{2\pi^2} - \frac{1}{4}\right] = L^2\left[\frac{1}{12} - \frac{1}{2\pi^2}\right]$$

$\Delta x \approx 0.18L$

$$\langle p \rangle = \sqrt{\frac{2}{L}} \int_0^L \sin\frac{\pi x}{L}\ \frac{\hbar}{i}\frac{d}{dx}\left(\sin\frac{\pi x}{L}\right) dx$$

$$= \sqrt{\frac{2}{L}}\left(\frac{\hbar}{i}\right)\left(\frac{\pi}{L}\right) \int_0^L dx \sin\frac{\pi x}{L} \cos\frac{\pi x}{L} = 0$$

$$\langle p^2 \rangle = 2m \langle E \rangle = \hbar^2 \langle k^2 \rangle = \hbar^2 \frac{\pi^2}{L^2}$$

$\Delta p = \hbar \pi / L$

$\Delta x \cdot \Delta p = \hbar \pi (0.18) = 0.57 \hbar > \hbar/2$

6.15

6.12 (a) the state is a superposition of energy eigenstates.

$$|\psi\rangle = \left(\frac{1+i}{2}\right)|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$$

where $\langle x|n\rangle = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ are the particle in a box wave functions.

These states are orthogonal:

$$\langle m|n\rangle = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = \delta_{mn}$$

$$|\psi(t)\rangle = \left(\frac{1+i}{2}\right) e^{-iE_1 t/\hbar}|1\rangle + \frac{1}{\sqrt{2}} e^{-iE_2 t/\hbar}|2\rangle$$

where $E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2} = 4E_1$

or, in terms of the wave functions:

$$\langle x|\psi(t)\rangle = \left(\frac{1+i}{2}\right)\exp\left(-\frac{i\hbar t\pi^2}{2mL^2}\right)\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

$$+ \frac{1}{\sqrt{2}}\exp\left(-\frac{4i\hbar t\pi^2}{2mL^2}\right)\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$$

(b) $|\Psi(t)\rangle = \left(\frac{1+i}{2}\right) e^{-iE_1 t/\hbar} |1\rangle + \frac{1}{\sqrt{2}} e^{-4iE_1 t/\hbar} |2\rangle$

$\langle E \rangle = \langle \Psi | \hat{H} | \Psi \rangle =$

$$\left[\left(\frac{1-i}{2}\right) e^{iE_1 t/\hbar} \langle 1| + \frac{1}{\sqrt{2}} e^{+4iE_1 t/\hbar} \langle 2|\right]$$

$$\times \left[\left(\frac{1+i}{2}\right) e^{-iE_1 t/\hbar} E_1 |1\rangle + \frac{1}{\sqrt{2}} e^{-i4iE_1 t/\hbar} 4E_1 |2\rangle\right]$$

$$= \frac{1}{2} E_1 + \frac{4E_1}{2} = \left(\frac{5}{2}\right) E_1$$

(c) the probability to measure the ground state energy E_1 is

$$|\langle 1|\Psi\rangle|^2 = \left|\frac{1+i}{2}\right|^2 = \frac{1}{2}$$

(d) $\langle \hat{x} \rangle = \langle \Psi(t) | \hat{x} | \Psi(t) \rangle$ is time dependent because $\langle 1 | \hat{x} | 2 \rangle$ is not zero. in this case

in terms of the matrix $\langle n | \hat{x} | m \rangle \equiv X_{nm} = X_{mn}$

$$\langle \hat{x} \rangle = \left[\left(\frac{1-i}{2}\right) e^{i\omega_1 t} \langle 1| + \frac{1}{\sqrt{2}} e^{4i\omega_1 t} \langle 2|\right] \quad \left(\omega_1 \equiv \frac{E_1}{\hbar}\right)$$

$$\cdot \hat{x} \left[\frac{1+i}{2} e^{-i\omega_1 t} |1\rangle + \frac{1}{\sqrt{2}} e^{-4i\omega_1 t} |2\rangle\right]$$

$$\langle \hat{X} \rangle = \frac{1}{2} X'_{11} + \frac{1}{2} X_{22}$$

$$+ \left(\frac{1-i}{2}\right) e^{-3i\omega_1 t} X_{12} + \left(\frac{1+i}{2}\right) e^{3i\omega_1 t} X_{12}^*$$

$$\langle \hat{X} \rangle = \frac{1}{2}(X_{11} + X_{22}) + X_{12}\left(\cos(3\omega_1 t) - \sin(3\omega_1 t)\right)$$

6.16 ~~6.13~~ Not assigned fall 2009

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) & 0 < x < L \\ 0 & x < 0,\ x > L \end{cases}$$

(a) to calculate the probability to be in the ground state of the new box of ~~length~~ 2L we calculate the integral:

$$\langle 1 | \psi \rangle = \int_0^L dx \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \sqrt{\frac{1}{L}} \sin\left(\frac{\pi x}{2L}\right)$$

where the limit are 0 to L because $\Psi(x,0)$ is zero elsewhere.

$$\langle 1, 2L | 1, L \rangle = \frac{\sqrt{2}}{L}\left(\frac{L}{\pi}\right) 2 \int_0^{\pi/2} dy \sin(2y) \sin(y)$$

$$= \frac{\sqrt{2}}{\pi} 4 \int_0^{\pi/2} dy \sin^2 y \cos y = \frac{\sqrt{2}}{\pi} \frac{4}{3} \left(\sin^3 y\right)\Big|_0^{\pi/2}$$

$$= \frac{4\sqrt{2}}{3\pi}$$

then $\quad |\langle 1|\Psi\rangle|^2 = \frac{32}{9\pi^2} = 0.36$

(b) Expand the state at $t=0$ in terms of energy eigenfunction:

$$|\Psi(0)\rangle = \sum_{n=1}^{\infty} |n\rangle\langle n|\Psi\rangle \equiv \sum_{n=1}^{\infty} A_n |n\rangle$$

then $$|\Psi(t)\rangle = \sum_{n=1}^{\infty} e^{-iE_n t/\hbar} |n\rangle\langle n|\Psi\rangle$$

$$\Psi(x,t) = \langle x|\Psi(t)\rangle = \sum_{n=1}^{\infty} e^{-iE_n t/\hbar} \langle x|n\rangle\langle n|\Psi\rangle$$

$$= \sum_{n=1}^{\infty} e^{-iE_n t/\hbar} \psi_n(x) A_n$$

where $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{2L}$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2m(2L)^2}$$

Clearly this state is not stationary.

6.18 6.15

(a) for an electron with $L = \mathring{A} = 10^{-8}\text{ cm} = 0.1\text{ nm}$

$$E_1 = \frac{\hbar^2\pi^2}{2mL^2} = \frac{(\hbar c)^2\pi^2}{2(mc^2)L^2}$$

with $\hbar c = 197\text{ eV}\cdot\text{nm}$

$mc^2 = 0.5\times10^6\text{ eV}$

$$E_1 = \frac{(197\text{ eV}\cdot\text{nm})^2\,\pi^2}{10^6\text{eV}\,(\frac{1}{10}\text{nm})^2} = (1.97\times\pi)^2\text{ eV} = \underline{38.3\text{eV}}$$

(b) for a proton in a nucleus, $L = 10\text{ fm}$

$M_pc^2 \cong 400\text{ MeV}$

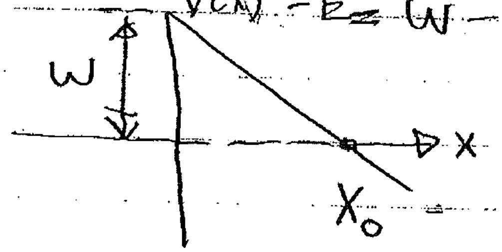
$\hbar c \cong 200\text{ MeV-fm}$

$$E_1 = \frac{(200\text{ MeV}\cdot\text{fm})^2\pi^2}{400\text{ MeV}\,(10\text{ fm})^2} = \pi^2\text{ MeV} \approx \underline{10\text{MeV}}$$

6.25

6.21

$$V(x) - E = W - \alpha x, \quad \alpha = e|E|$$



the width of the barrier is obtained from

$$W - \alpha X_0 = 0 \quad \Rightarrow X_0 = \frac{W}{\alpha}$$

Then the integral for the tunneling factor is:

$$I = \int_0^{X_0} dx \sqrt{\frac{2m}{\hbar^2}(W - \alpha x)}$$

$$= \frac{\hbar^2}{2m\alpha} \int_0^{2mW/\hbar^2} dy \sqrt{y} = \frac{\hbar^2}{2m\alpha}\left(\frac{2}{3}\right) y^{3/2} \Big|_0^{2mW/\hbar^2}$$

$$= \frac{\hbar^2}{2m\alpha}\left(\frac{2}{3}\right)\left(\frac{2mW}{\hbar^2}\right)^{3/2} = \frac{2\sqrt{2m}}{3\hbar\alpha} W^{3/2}$$

then $T = \exp(-2I) = \exp\left(-\dfrac{4\sqrt{2m}\, W^{3/2}}{3\hbar e|E|}\right)$