Physics 491
m. Gold

HW # 11 Solutions

7.2

$$[\hat{x}] = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ & \vdots & & & \end{pmatrix}$$

$$[\hat{p}] = -i\sqrt{\frac{m\omega\hbar}{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & -\sqrt{3} & 0 & \\ & & \vdots & & \end{pmatrix}$$

$$[\hat{x}][\hat{p}] = -i\left(\frac{\hbar}{2}\right) \begin{pmatrix} -1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \\ & \vdots & & & \end{pmatrix} = i\frac{\hbar}{2}[1]$$

$$[\hat{p}][\hat{x}] = -i\left(\frac{\hbar}{2}\right) \begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ & \vdots & & & \end{pmatrix} = -i\frac{\hbar}{2}[1]$$

$$[\hat{x}][\hat{p}] - [\hat{p}][\hat{x}] = [\hat{x}, \hat{p}] = i\hbar[1]$$

7.4 $\hat{a}|0\rangle = 0 \Rightarrow \langle p|\hat{a}|0\rangle = 0$

write $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$ to get

$$0 = \langle p|\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)|0\rangle$$

$$= \langle p|\hat{x}|0\rangle + \frac{i}{m\omega}p\langle p|0\rangle$$

$$= i\hbar\frac{\partial}{\partial p}\langle p|0\rangle + \frac{i}{m\omega}p\langle p|0\rangle$$

$$\frac{\partial}{\partial p}\langle p|0\rangle = -\frac{p}{\hbar m\omega}\langle p|0\rangle$$

$$\ln\langle p|0\rangle = -\frac{1}{2}\frac{p^2}{\hbar m\omega} + \text{Constant}$$

$$\langle p|0\rangle = N\exp\left(-\frac{p^2}{2\hbar m\omega}\right)$$

$$\langle p|0\rangle = \left(\frac{1}{\hbar m\omega\pi}\right)^{1/4}\exp\left(-\frac{p^2}{2\hbar m\omega}\right)$$

7.7

$$E_N = \hbar\omega\left(N+\tfrac{1}{2}\right)$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$$

where ϕ is an unknown phase

$$\langle \hat{p}_x\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\,\langle\psi|(\hat{a}-\hat{a}^+)|\psi\rangle$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\langle 0| + e^{-i\phi}\langle 1|\right)\left(a-a^+\right)\left(|0\rangle + e^{i\phi}|1\rangle\right)$$

$\langle 0|\hat{a}|1\rangle = 1 \qquad \langle 1|\hat{a}|0\rangle = 0$

$\langle 1|\hat{a}^+|0\rangle = 1 \qquad \langle 0|a^+|1\rangle = 0$

$$\langle \hat{p}_x\rangle = \sqrt{\frac{m\omega\hbar}{2}}\,\frac{2}{2i}\left(e^{i\phi}-e^{-i\phi}\right) = \sqrt{\frac{m\omega\hbar}{2}}\,\sin\phi$$

$$\langle p_x\rangle = \sqrt{\frac{m\omega\hbar}{2}} \Rightarrow \sin\phi = 1,\ \underline{\phi = \pi/2}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\frac{\pi}{2}}|1\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\omega t}{2}}|0\rangle + e^{i\pi/2}e^{-i\omega t\frac{3}{2}}|1\rangle\right)$$

$$= \frac{e^{-i\omega t/2}}{\sqrt{2}}\left(|0\rangle + e^{i\left(\frac{\pi}{2}-\omega t\right)}|1\rangle\right)$$

$$\langle p_x\rangle = \sqrt{\frac{m\omega\hbar}{2}}\,\sin\left(\frac{\pi}{2}-\omega t\right) = \sqrt{\frac{m\omega\hbar}{2}}\,\cos(\omega t)$$

7.8 At the classical turning point (x_0) the energy is purely potential.

$$E_0 = \frac{\hbar\omega}{2} = \frac{1}{2} m \omega^2 x_0^2$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

for $m = 10^6\ g$ $\quad \omega = 2\pi \times 10^3\ s^{-1}$ $\quad \hbar = 1.05 \times 10^{-27}$ erg-s

$[\text{erg-s}] = [g\ cm^2/s]$

$$x_0 = \sqrt{\frac{1.05 \times 10^{-27}}{10^6 \times 2\pi \times 10^3}}\ cm = 0.4 \times 10^{-18}\ cm = \underline{0.4 \times 10^{-5}\ fm}$$

Infinitessimal compared to the size of the proton!

(b) $\quad x_n = \sqrt{2n}\, x_0 \quad$ for $n \gg 1$, $E_n \hat{=} n\hbar\omega$

so if $n \to n+1$, $\quad x_{n+1} - x_n = \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{2}\, x_0$

with $\sqrt{n+1} = \sqrt{n}\left(1 + \frac{1}{n}\right)^{1/2} \cong \sqrt{n}\left(1 + \frac{1}{2}\frac{1}{n}\right)$

$$x_{n+1} - x_n = x_0 \frac{1}{\sqrt{2n}}$$

This is the change in length of $\frac{1}{2}$ of the bar. The total change in length is then

$$\Delta x = 2(x_{n+1} - x_n) = \sqrt{\frac{2}{n}}\, x_0$$

(c) its nice to remember that kT @ 300k (room temperature) is $\approx$ 1/40 eV.

$$\frac{1}{2}kT\Big|_{T=1k} = \left(\frac{1}{40}\text{eV}\right)\frac{1}{2}\left(\frac{1}{300}\right) = 4\times10^{-5}\,\text{eV} = N\hbar\omega$$

$$N = \frac{4\times10^{-5}\,\text{eV}}{\hbar(2\pi)\,10^{3}\,s^{-1}} = \frac{4\times10^{-5}\,\text{eV}}{(2\pi)(6.582\times10^{-16}\,\text{eV}\cdot s)\,10^{3}\,s^{-1}}$$

$\hbar$ 6.582x10^{-16} eV-s

$$= \frac{40}{2\pi\,(6.582)}\times10^{7} \cong \underline{10^{7}}$$

<u>Conclusion</u>: you could not possibly detect the absorption of a <u>single</u> graviton.

7.12

$$\psi(x) = \langle w|x\rangle = Ce^{-ax^2}$$

let $y = \sqrt{\frac{m\omega}{\hbar}}\, x$

$$\psi(y) = Ce^{-\alpha y^2} \qquad \alpha \equiv \left(\frac{\hbar}{m\omega} a\right)$$

$$\psi'' + (\varepsilon - y^2)\psi = 0 \qquad \varepsilon \equiv 2E/\hbar\omega$$

$$\psi' = -2\alpha y \psi$$
$$\psi'' = -2\alpha\psi + 4\alpha^2 y^2 \psi$$

$$(4\alpha^2 y^2 - y^2)\psi + (\varepsilon - 2\alpha)\psi = 0$$

$$4\alpha^2 = 1 \Rightarrow \alpha = \frac{1}{2} \Rightarrow a = \left(\frac{m\omega}{2\hbar}\right)$$

$$\varepsilon = 2\alpha = 1$$
$$E = \hbar\omega/2$$ the ground state.

7.13

$$p = 2\int_{x_0}^{\infty} |\psi_0|^2 dx \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\langle x|0\rangle = \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

$$p = 2\int_{x_0}^{\infty} |\psi_0|^2 dx = 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{x_0}^{\infty} e^{-\frac{m\omega x^2}{\hbar}} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-u^2} du$$

the error function is defined as,

$$erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

So that

$$P = 1 - erf(1) = 1 - 0.84270$$

$$P = 0.157$$

12-9

Time evolution of Gaussian free particle state

$$\psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$$

$$\psi(x,t) = A(t)\int dx'\, e^{iS[\bar{x}]/\hbar}\, \psi(x',0)$$

free particle:

$$A(t) = \sqrt{\frac{m}{2\pi i \hbar t}} \qquad S[\bar{x}] = \frac{1}{2}\frac{m}{t}(x-x')^2$$

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i \hbar t}} \int dx'\, \exp\left[\frac{im}{2\hbar t}(x-x')^2 - ax'^2\right]$$

$$[\ ] = \frac{im}{2\hbar t}x^2 - \frac{im}{\hbar t}xx' + \underbrace{\left(\frac{im}{2\hbar t} - a\right)}_{\frac{im}{2\hbar t}\left(1+\frac{2ia\hbar t}{m}\right) \equiv \frac{im}{2\hbar t}\alpha} x'^2$$

$$= \frac{im}{2\hbar t}\left(x^2 - 2xx' + \alpha x'^2\right)$$

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{m}{2\pi i \hbar t}}\, e^{imx^2/2\hbar t}\int dx'\, \exp \frac{im}{2\hbar t}(\alpha x'^2 - 2xx')$$

$$I(c,b) = \int dx\, e^{-cx^2+bx} = \exp\left(\frac{b^2}{4c}\right)\sqrt{\frac{\pi}{c}}$$

$$\text{with } b = \frac{-im}{\hbar t} \qquad c = \frac{-im}{2\hbar t}\alpha$$

$$\frac{b^2}{4c} = -\left(\frac{m}{\hbar t}\right)^2 x^2 \frac{1}{4}\frac{1}{\frac{-im}{2\hbar t}\alpha} = \frac{-ix^2}{4}\frac{1}{\frac{\alpha}{2}}\left(\frac{m}{\hbar t}\right) = \frac{-ix^2}{\alpha}\left(\frac{m}{2\hbar t}\right)$$

then

$$\psi(x,t) = D \; \exp\left[i x^2 \frac{m}{2\hbar t}\left(1 - \frac{1}{\alpha}\right)\right]$$

$$D = \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i \hbar t}} \sqrt{\frac{\pi}{\frac{-im}{2\hbar t}\alpha}} = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{\alpha}}$$

$$[\;] = i x^2 \frac{m}{2\hbar t}\left(\frac{\alpha - 1}{\alpha}\right) = i x^2 \frac{m}{2\hbar t}\left(\frac{2ia\hbar t}{m}\right)\frac{1}{\alpha} = -a\frac{x^2}{\alpha}$$

$$\psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{\alpha}} e^{-ax^2/\alpha} \quad ; \; \alpha = 1 + \frac{2ia\hbar t}{m}$$

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8.3 Since the Lagrangian is quadratic, we know that the path integral becomes

$$\langle x_b, t_b | x_a, t_a \rangle = e^{iS_{cl}/\hbar} F(t_b - t_a)$$

We need to calculate the action along the classical path, S_{cl}.

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right)$$

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

$$S_{cl} = \int_{t_a}^{t_b} L(\dot{x}, x, t)\,dt = \int_0^T L(\dot{x}', x', t')\,dt'$$

where $x'(t') = x_a \cos\omega t' + \frac{v_0}{\omega} \sin\omega t'$

$$x'(0) = x_a$$

$$x'(T) = x_b = x_a + \frac{v_0}{\omega}\sin\omega T$$

$$x'(t') = x_a \cos\omega t' + \left(\frac{x_b - x_a \cos\omega T}{\sin\omega T}\right)\sin\omega t'$$

now we drop the prime and define $C \equiv \cos\omega T$
$S \equiv \sin\omega T$, $B \equiv (x_b - x_a C)/S$

$$\dot{x} = -\omega x_a \sin\omega t + \omega B \cos\omega t$$

$$L = \frac{m\omega^2}{2}\left(x_a \sin\omega t - B\cos\omega t\right)^2 - \frac{m\omega^2}{2}\left(x_a \cos\omega t + B\sin\omega t\right)^2$$

$$= \frac{m\omega^2}{2}\Big[\left(B^2 - x_a^2\right)\left(\cos^2\omega t - \sin^2\omega t\right) - 4Bx_a \sin\omega t\cos\omega t\Big]$$

$$L = \frac{m\omega^2}{2}\left[\left(B^2 - x_a^2\right)\cos 2\omega t - 2Bx_a \sin 2\omega t\right]$$

Then Scl is,

$$S_{cl} = \int_0^T L\,dt = \frac{m\omega^2}{2}\left(\frac{1}{2\omega}\right)\left(B^2 - X_a^2\right)\sin 2\omega T$$

$$+ \frac{m\omega^2}{2}\left(2BX_a\right)\frac{1}{2\omega}\left(\cos 2\omega T - 1\right)$$

$$= \frac{m\omega}{4}\left[\left(B^2 - X_a^2\right)\underbrace{\sin 2\omega T}_{2CS} + 2BX_a\left(\underbrace{\cos 2\omega T - 1}_{-2S^2}\right)\right]$$

$$S_{cl} = \frac{m\omega}{2} S\left[\left(B^2 - X_a^2\right)C - 2BX_a S\right]$$

$$2BX_a S = 2X_a\left(X_b - X_a C\right)$$

$$B^2 - X_a^2 = \frac{1}{S^2}\left(X_b^2 - 2X_b X_a C + X_a^2\left(C^2 - S^2\right)\right)$$

$$= \frac{1}{S^2}\left(X_b^2 + X_a^2\right) - \frac{2X_b X_a C}{S^2} - 2X_a^2$$

$$S_{cl} = \frac{m\omega}{2} S\left[\frac{C}{S^2}\left(X_b^2 + X_a^2\right) - 2X_b X_a \frac{C^2}{S^2} - 2X_a^2 C\right.$$

$$\left. - 2X_b X_a + 2X_a^2 C\right]$$

$$= \frac{m\omega}{2} S\left[\frac{C}{S^2}\left(X_b^2 + X_a^2\right) - \frac{2X_b X_a}{S^2}\right]$$

$$\boxed{S_{cl} = \frac{m\omega}{2\sin\omega T}\left[\cos\omega T\left(X_b^2 + X_a^2\right) - 2X_b X_a\right]}$$

8.5 Classical mechanics is expected to be valid if $S_{cl}/\hbar >>> 1$.

We can take the action to be approximately the free particle action,

$$S_{cl} \sim \frac{1}{2} m \frac{(\Delta x)^2}{\Delta t} = \frac{1}{2} m v \Delta x$$

(a) for an electron with $v/c \approx \alpha$ and $\Delta x = 0.05\,\text{nm}$

$$\frac{S_{cl}}{\hbar} = \frac{\frac{1}{2} m c^2 \left(\frac{v}{c}\right) \Delta x}{\hbar c} = \frac{\frac{1}{2}\left(5 \times 10^5\,\text{eV}\right)\frac{1}{137}(.05\,\text{nm})}{200\,\text{eV-nm}}$$

$$\frac{S_{cl}}{\hbar} \sim 1$$

$\therefore$ Q.M. effects are important

(b) for an electron of the same speed traveling $1\,\text{cm} = 10^{+7}\,\text{nm}$, $S_{cl}/\hbar \sim 10^{7}$

and we can expect an approximately classical trajectory.

8b $V = qV_i$

$$L = \frac{1}{2}m\dot{x}^2 - qV_i \qquad V_i \text{ const.}$$

$$S = \frac{1}{2}m\frac{(\Delta x)^2}{\Delta t} - qV_i\Delta t$$

extra factor is independent of path and can be taken outside functional integral, so that

$$\langle b|a\rangle = e^{-\frac{i}{\hbar}qV_iT}\langle b|a\rangle_0$$

where $\langle b|a\rangle_0$ is the free particle propagator.

$$A = \langle b|a\rangle_1 + \langle b|a\rangle_2 = \langle b|a\rangle_0\left(e^{-\frac{i}{\hbar}qV_1T} + e^{-\frac{i}{\hbar}qV_2T}\right)$$

$$P = |A|^2 \propto \left|1 + e^{-\frac{i}{\hbar}q\Delta VT}\right|^2 \qquad \Delta V = V_2 - V_1$$

$$\boxed{P = \frac{1}{2}\left(1 + \cos\left(q\Delta VT/\hbar\right)\right)}$$

Classically, the particle in a constant potential experiences no force and therefore one would expect (classically) no effect.