HW #2 Problems
Intermediate Quantum 491

---

1. Consider the matrix that rotates a Euclidean vector in the x-y plane,

$$\hat{R}^E(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

   Find the (complex) eigenvalues and corresponding eigenvectors. Write the simlarity transformation matrix $\hat{S}$ with eigenvectors as columns and show explicitly that $\hat{S}^\dagger \hat{R}^E S$ is diagonal. Show that S is unitary, that is $\hat{S}^\dagger S = \hat{I}$ where $\hat{I}$ is the identity matrix. Find the transformed eigenvectors in the new basis where $\hat{R}^E$ is diagonal.

2. Show that plausible representations of the Dirac delta function are,

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon}$$

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$$

3. Show that $\delta(cx) = \delta(x)/|c|$ where c is a constant. note that $\delta(-x) = \delta(x)$.

***Problem 2.1** Prove the following theorems:

(a) For normalizable solutions, the separation constant E must be *real*. *Hint*: Write E (in Equation 2.6) as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that if Equation 1.20 is to hold for all t, Γ must be zero.

(b) ψ can always be taken to be *real* (unlike Ψ, which is necessarily complex). *Note*: This doesn't mean that every solution to the time-independent Schrödinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So in Equation 2.14 you *might as well* stick to ψ's that are real. *Hint*: If $\psi(x)$ satisfies the time-independent Schrödinger equation for a given E, so too does its complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i(\psi - \psi^*)$.

(c) If $V(x)$ is an *even* function [i.e., $V(-x) = V(x)$], then $\psi(x)$ can always be taken to be either even or odd. *Hint*: If $\psi(x)$ satisfies the time-independent Schrödinger equation for a given E, so too does $\psi(-x)$, and hence also the even and odd linear combinations $\psi(x) \pm \psi(-x)$.

2.6
$\Psi(x,t) = e^{-iEt/\hbar}\psi(x)$

***Problem 2.5** Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, σ_x, and σ_p, for the nth stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

****Problem 2.6** A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)].$$

(a) Normalize $\Psi(x,0)$. (That is, find A. This is very easy if you exploit the orthonormality of ψ_1 and ψ_2. Recall that, having normalized Ψ at $t = 0$, you can rest assured that it *stays* normalized—if you doubt this, check it explicitly after doing part b.)

(b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. (Express the latter in terms of sinusoidal functions of time, eliminating the exponentials with the help of **Euler's formula**: $e^{i\theta} = \cos\theta + i\sin\theta$.) Let $\omega \equiv \pi^2\hbar/2ma^2$.

(c) Compute $\langle x\rangle$. Notice that it oscillates in time. What is the frequency of the oscillation? What is the amplitude of the oscillation? (If your amplitude is greater than $a/2$, go directly to jail.)

(d) Compute $\langle p\rangle$. (As Peter Lorre would say, "Do it ze *kveek* vay, Johnny!")

(e) Find the expectation value of H. How does it compare with E_1 and E_2?

(f) A *classical* particle in this well would bounce back and forth between the walls. If its energy is equal to the expectation value you found in (e), what is the frequency of the classical motion? How does it compare with the quantum frequency you found in (c)?

***Problem 2.10** The wave function (Equation 2.14) has got to be normalized; given that the ψ_n's are orthonormal, what does this tell you about the coefficients c_n? *Answer*:

$$\sum_{n=1}^{\infty} |c_n|^2 = 1. \qquad [2.34]$$

(In particular, $|c_n|^2$ is always ≤ 1.) Show that

$$\langle H\rangle = \sum_{n=1}^{\infty} E_n |c_n|^2. \qquad [2.35]$$

Incidentally, it follows that $\langle H\rangle$ is constant in time, which is one manifestation of **conservation of energy** in quantum mechanics.

$\Psi(x,t) = \sum c_n \psi_n(x)\, e^{-iE_nt/\hbar}$ (2.14)

***Problem 2.22** A free particle has the initial wave function

$$\Psi(x, 0) = Ae^{-ax^2},$$

where A and a are constants (a is real and positive).

(a) Normalize $\Psi(x, 0)$.

(b) Find $\Psi(x, t)$. *Hint:* Integrals of the form

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\, dx$$

can be handled by "completing the square." Let $y \equiv \sqrt{a}[x + (b/2a)]$, and note that $(ax^2 + bx) = y^2 - (b^2/4a)$. *Answer:*

$$\Psi(x, t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1 + (2i\hbar at/m)}}.$$

(c) Find $|\Psi(x, t)|^2$. Express your answer in terms of the quantity $w \equiv \sqrt{a/[1 + (2\hbar at/m)^2]}$. Sketch $|\Psi|^2$ (as a function of x) at $t = 0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$ as time goes on?

(d) Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, σ_x, and σ_p. *Partial answer:* $\langle p^2\rangle = a\hbar^2$, but it may take some algebra to reduce it to this simple form.

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

***Problem 2.26** Consider the *double* delta-function potential

$$V(x) = -\alpha[\delta(x + a) + \delta(x - a)],$$

where α and a are positive constants.

(a) Sketch this potential.

(b) How many bound states does it possess? Find the allowed energies, for $\alpha = \hbar^2/ma$ and for $\alpha = \hbar^2/4ma$, and sketch the wave functions.

Problem 2.30 The Dirac delta function can be thought of as the limiting case of a rectangle of area 1, as the height goes to infinity and the width goes to zero. Show that the delta-function well (Equation 2.96) is a "weak" potential (even though it is

$V(x) = -\alpha\,\delta(x)$ Eq. 2.96

infinitely deep), in the sense that $z_0 \to 0$. Determine the bound-state energy for the delta-function potential, by treating it as the limit of a finite square well. Check that your answer is consistent with Equation 2.111. Also show that Equation 2.151 reduces to Equation 2.123 in the appropriate limit. $\longrightarrow$ next page

***Problem 2.34** Construct the S-matrix for scattering from a delta-function well (Equation 2.96). Use it to obtain the bound state energy, and check your answer against Equation 2.111.

2.111 $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$; $E = \frac{m\alpha^2}{2\hbar^2}$

2.123 $R = [1 + (2\hbar^2/E/m\alpha^2)]^{-1}$; $T = [1 + (m\alpha^2/2\hbar^2 E)]^{-1}$

Problem 2.46 Consider the potential

$$V(x) = \begin{cases} \infty, & \text{if } x < 0, \\ \alpha\delta(x-a), & \text{if } x \geq 0, \end{cases}$$

where a and α are positive real constants with the appropriate units (see Figure 2.18). A particle starts out in the "well" ($0 < x < a$), but because of tunneling its wave function gradually "leaks" out through the delta-function barrier.

(a) Solve the (time-independent) Schrödinger equation for this potential; impose appropriate boundary conditions, and determine the "energy", E. (An implicit equation will do.)

(b) I put the word "energy" in quotes because you'll notice that it is a *complex number*! How do you account for this, in view of the theorem you proved in Problem 2.1a?

(c) Writing $\cancel{E = E_0 + i\Gamma}$ (with E_0 and Γ real), calculate (in terms of Γ) the characteristic time it takes the particle to leak out of the well (that is, the time it takes before the probability is $1/e$ that it's still in the region $0 < x < a$).

E=E_0 - iΓ/2

2.151

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

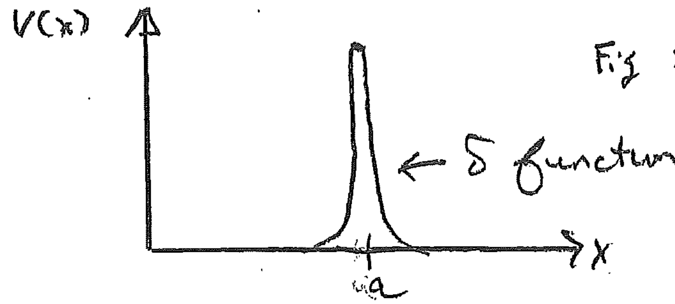

Fig 2.18