Fall 2015

Physics 491

HW # 4 Solutions

2.1 Prove that $\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x$

This is an exercise in series convergence in real analysis. See any real analysis text for a rigorous proof. You could take as a definition of Euler's constant

$$e \equiv \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N$$

from which the result follows. Here is my "physicist's" proof. Binomial expansion of $\left(1+\frac{x}{N}\right)^N$:

$$\left(1+\frac{x}{N}\right)^N = \sum_{m=0}^{N} \frac{x^m}{m!}\left(\frac{N!}{N^m(N-m)!}\right)$$

$$\lim_{N\to\infty} \frac{N!}{N^m(N-m)!} = \lim_{N\to\infty} \frac{N(N-1)\cdots(N-m)}{N^m} = 1$$

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = \sum_{m=0}^{\infty}\frac{x^m}{m!} = e^x$$

2.3 $\hat{R}_z(\phi) = e^{-i\phi \hat{S}_z/\hbar}$

So $\hat{R}_z(\phi)|\pm z\rangle = e^{\mp i\phi/2}|\pm z\rangle$

thus $\left[R_z(\phi)\right] = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$

$$\left[R^\dagger\right]_{ij}\left[R_{ji}\right] = \begin{pmatrix} e^{+i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{R}^\dagger \hat{R} = \hat{1}$$

4 The components of $|\pm x\rangle$ in the $|\pm y\rangle$ basis are the expansion coefficients,

$$|\pm x\rangle = |+y\rangle\langle +y|\pm x\rangle + |-y\rangle\langle -y|\pm x\rangle$$

$$|\pm x\rangle \longleftrightarrow \begin{pmatrix} \langle +y|\pm x\rangle \\ \langle -y|\pm x\rangle \end{pmatrix}_y$$

Use $|\pm x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle \pm |-z\rangle\right)$ and

$$\langle \pm y| = \frac{1}{\sqrt{2}}\left(\langle +z| \mp i\langle -z|\right)$$

to get $\langle +y|+x\rangle = \frac{1}{2}(1-i)$ and $\langle -y|+x\rangle = \frac{1}{2}(1+i)$

and $|+x\rangle \rightarrow \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$; $|-x\rangle \rightarrow \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$

$\boxed{2.4}$

$$|\pm x\rangle = |+y\rangle\langle +y|\pm x\rangle + |-y\rangle\langle -y|\pm x\rangle$$

$$|\pm x\rangle \xrightarrow[y\,\text{basis}]{} \begin{pmatrix} \langle +y|\pm x\rangle \\ \langle -y|\pm x\rangle \end{pmatrix}$$

we can calculate these in the z basis:

$$|\pm x\rangle \xrightarrow[z\,\text{basis}]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \quad ; \quad |\pm y\rangle \xrightarrow[z]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

$$\langle +y|\pm x\rangle = \frac{1}{2}(1, i^*)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{2}(1 \mp i)$$

$$\langle -y|\pm x\rangle = \frac{1}{2}(1, -i^*)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{2}(1 \pm i)$$

Alternatively, we could use the basis transformation:

$$(|+x\rangle, |-x\rangle) = (|+z\rangle, |-z\rangle)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$(|+y\rangle, |-y\rangle) = (|+z\rangle, |-z\rangle)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$(|+x\rangle, |-x\rangle)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (|+y\rangle, |-y\rangle)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$(|+x\rangle, |-x\rangle) = (|+y\rangle, |-y\rangle)\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= (|+y\rangle, |-y\rangle)\frac{1}{2}\begin{pmatrix} 1-i, & 1+i \\ 1+i, & 1-i \end{pmatrix}$$

2.5 Find the matrix representation of $\hat{S}_z$ in the y-basis. $|+y\rangle \rightarrow \begin{pmatrix}1\\0\end{pmatrix}; |-y\rangle \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$$\hat{S}_z |\pm y\rangle = \hat{S}_z \frac{1}{\sqrt{2}}\left(|+z\rangle \pm i\,|-z\rangle\right)$$

$$= \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(|+z\rangle \mp i\,|-z\rangle\right) = \frac{\hbar}{2}|\mp y\rangle$$

thus

$$\left[\hat{S}_z\right] = \begin{bmatrix} \langle +y|\hat{S}_z|+y\rangle & \langle +y|\hat{S}_z|-y\rangle \\ \langle -y|\hat{S}_z|+y\rangle & \langle -y|\hat{S}_z|-y\rangle \end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\langle \hat{S}_z \rangle_{-y} = \langle -y|\hat{S}_z|-y\rangle = (0,1)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

HW #4 Solutions

2.8 $|\psi\rangle \xrightarrow[+z]{} \frac{1}{\sqrt{5}}\begin{pmatrix} i \\ 2 \end{pmatrix}$

$\langle\psi|\psi\rangle = \frac{1}{5}(-i, 2)\begin{pmatrix} i \\ 2 \end{pmatrix} = 1$

Probability for $\hat{S}_x$ to be $+\frac{\hbar}{2}$ is $|\langle +x|\psi\rangle|^2$

$|+x\rangle \xrightarrow[+z]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\langle +x|\psi\rangle = \frac{1}{\sqrt{10}}(1, 1)\begin{pmatrix} i \\ 2 \end{pmatrix} = \frac{1}{\sqrt{10}}(i+2)$

$|\langle +x|\psi\rangle|^2 = \frac{1}{10}(-i+2)(i+2) = \frac{5}{10} = \frac{1}{2}$

similarly, $|+y\rangle \xrightarrow[+z]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\langle +y|\psi\rangle = \frac{1}{\sqrt{10}}(1, -i)\begin{pmatrix} i \\ 2 \end{pmatrix} = \frac{1}{\sqrt{10}}(-i)$

$|\langle +y|\psi\rangle|^2 = \frac{1}{10}$

Hw #4 Solutions

(2.10) $$\left(|+x\rangle, |-x\rangle\right) = \left(|+z\rangle, |-z\rangle\right) \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}_{\hat{S}}$$

$$\left[\hat{S}_x\right]_{+x} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\left[\hat{S}_x\right]_{+z} = \hat{S}^{+}\hat{S}_x\hat{S} = \frac{\hbar}{2}\left(\frac{1}{2}\right)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\frac{1}{2}\right)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$