Physics 491
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Lecture 10: Rotations & Spin

Operator for rotation of spinor about arbitrary axis: $R^s(\phi\hat{n})$?

<u>Theorem</u>: Any unitary operator $\hat{U}$ $(\hat{U}^+\hat{U} = \hat{I})$ can be written as exponential of Hermitian operator $(\hat{A} = \hat{A}^+)$.

$$\hat{U} = \exp(-i\hat{A}) \equiv 1 - i\hat{A} - \tfrac{1}{2}\hat{A}^2 + \ldots$$

defined by power series

<u>Baker-Hausdorff Lemma</u>: $\hat{A}, \hat{B}$ any operators,

Commutator $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

If $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$

then $e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{\frac{1}{2}[\hat{A}, \hat{B}]}$

Then for <u>Hermitian</u> $\hat{A}$, $\hat{U}^+\hat{U} = \exp(i\hat{A})\exp(-i\hat{A}) =$

since $[\hat{A}, \hat{A}] = 0$ $\qquad \exp[i(\hat{A} - \hat{A})] = \hat{I}$

Thus we may write general rotation operator as

$$\hat{R}(\phi\hat{z}) = e^{-i\phi\hat{J}_z/\hbar}$$

where $\hbar$ gives $\hat{J}_z$ units of angular momentum.

$\hat{J}_z/\hbar$ is the <u>generator</u> of infinitessimal rotations:

$$\frac{\hat{J}_z}{\hbar} = i \frac{d}{d\phi}\left(\hat{R}_z(\phi)\right)\Big|_{\phi=0}$$

Finite rotation is product of infinitessimal rotations in the limit

$$\hat{R}(\phi\hat{z}) = \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\hat{J}_z\left(\frac{\phi}{N}\right)\right]^N$$

$$= e^{-i J_z \phi/\hbar}$$

note that successive rotations about the same axis commute,

$$\hat{R}(\phi_1\hat{z})\hat{R}(\phi_2\hat{z}) = \hat{R}(\phi_2\hat{z})\hat{R}(\phi_1\hat{z}) = \hat{R}((\phi_1+\phi_2)\hat{z})$$

Rotations about <u>different</u> directions <u>do not commute</u>. <u>Algebra</u> <u>of generators</u> <u>defines</u> <u>group</u>.

$$\left[\frac{\hat{J}_x}{\hbar}, \frac{\hat{J}_y}{\hbar}\right] = i\,\frac{\hat{J}_z}{\hbar} \quad \text{and cyclic}$$

or $\hat{x}=1$, $\hat{y}=2$, $\hat{z}=3$ notation:

$$\left[\frac{\hat{J}_i}{\hbar}, \frac{\hat{J}_j}{\hbar}\right] = i\sum_k \varepsilon_{ijk}\frac{\hat{J}_k}{\hbar}$$

$\hat{R}^s(\phi\hat{z}) = e^{-i\phi\hat{S}_z/\hbar} = e^{-i\phi\hat{\sigma}_z/2}$ generator $\frac{\hat{\sigma}_z}{2}$

Spinor rotations

Rotations & generators are represented by 2×2 rotation matricies in some basis.

$|+z\rangle$ is invariant under $\hat{R}^s(\phi\hat{z})$ (spinor), thus

$$\hat{R}^s(\phi\hat{z})|+z\rangle = e^{-ia\phi}|+z\rangle$$

and $$\hat{R}^s(\phi\hat{z})|-z\rangle = e^{-ib\phi}|-z\rangle$$

Q.M. says we can have a phase! Since only phase differnce is observable, we can choose $b=-a$.

For infinitessimal ϕ,*

$$\frac{\hat{S}_z}{\hbar}|\pm z\rangle = \pm a|\pm z\rangle \quad \text{eigenvalue } \pm a$$

Knowing $|\pm x\rangle$, $|\pm y\rangle$ in the $+z$-basis and

$$R^s(\tfrac{\pi}{2}\hat{z})|\pm x\rangle = |\pm y\rangle$$

this fixes a: $|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle)$

$$|+y\rangle = R^s(\tfrac{\pi}{2}\hat{z})|+x\rangle = \frac{1}{\sqrt{2}}\left(\hat{R}^s_z|+z\rangle + \hat{R}^s_z|-z\rangle\right) = \frac{1}{\sqrt{2}}e^{-ia\frac{\pi}{2}}\left(|+z\rangle + e^{+ia\pi}|-z\rangle\right)$$

So $e^{ia\pi}=i$ and $a=\frac{1}{2}$. Choose to ignore overall phase in definition of $|\pm x\rangle$.

* I use $\vec{S}$ for spin ½, $\vec{J}$ for generall angular momentum operator

thus $\frac{\hat{S}_z}{\hbar}$ eigenvalues are $\pm\frac{1}{2}$ (spin-1/2)

with $\hat{S}_i = \frac{\hbar}{2}\hat{\sigma}_i$ ($\hat{\sigma}_i$ Pauli matrices)

$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$; $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

We can easily show

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\sum_k \epsilon_{ijk} \frac{\sigma_k}{2}$$

Spinor generator algebra <u>same</u> as 3x3 rotation generator algebra. ∴ <u>same group</u>

Group is $\boxed{SU(2)}$ defining representation is determinant +1, unitary, 2x2 matrices.

Be careful not to confuse group elements w/ generators. Pauli matrices are Hermitian, traceless, determinant −1.

<u>Note:</u> $\hat{R}^s(2\pi\hat{z})|\pm z\rangle = e^{\pm i\pi}|\pm z\rangle = -|\pm z\rangle$

This phase is physical and has been measured by interference experiment (Townsend Ch 4.)

Finite Spin Rotations

Since $\sigma_z^{\,2} = \hat{I}$ $\quad [\hat{R}^s(\phi\hat{z})]^{z\text{ basis}} =$

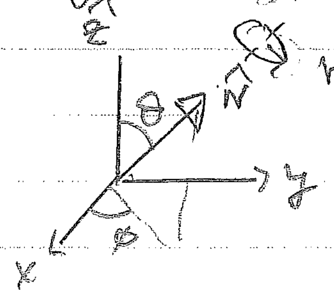
$$e^{-i\phi\hat{\sigma}_z/2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\phi\frac{\hat{\sigma}_z}{2}\right)^n$$

$$= 1 + \left(-i\phi\frac{\hat{\sigma}_z}{2}\right) + \frac{1}{2!}\left(-i\phi\frac{\hat{\sigma}_z}{2}\right)^2 + \frac{1}{3!}\left(-i\phi\frac{\hat{\sigma}_z}{2}\right)^3 + \cdots$$

$$= \cos\frac{\phi}{2}\hat{I} - i\sin\frac{\phi}{2}\hat{\sigma}_z = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$$

More generally, for $\hat{n} = \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}$

rotate about $\hat{n}$ by α



$$R^s(\alpha\hat{n}) = \cos\frac{\alpha}{2}\hat{I} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\alpha}{2}$$

$$\vec{\sigma} \equiv \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$$

Important that components of $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ transform as Euclidean vector!

To prove, use $(\vec{\sigma}\cdot\hat{n})^2 = \hat{I}$.

This can be proved from

$$\sigma_i \sigma_j + \sigma_j \sigma_i \equiv \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

"anti-commutator"

On H.W. you will find an overall phase factor for $|+n\rangle$

$$|+n\rangle = \cos\tfrac{\theta}{2}\, e^{-i\phi/2}\,|+z\rangle + \sin\tfrac{\theta}{2}\, e^{+i\frac{\phi}{2}}\,|-z\rangle$$

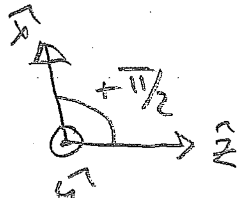
<u>Example</u> $\left[R^S(\theta\hat{y})\right]^{z} = \cos\frac{\theta}{2}\hat{I} - i\hat{\sigma}_y \sin\theta/2$

$-i\sigma_y = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ giving

$$\left[R^S(\theta\hat{y})\right]^{z} = \begin{pmatrix} \cos\theta/2, & -\sin\theta/2 \\ \sin\theta/2, & \cos\theta/2 \end{pmatrix}$$

check that det $=+1$

$\left[R^S(\tfrac{\pi}{2}\hat{y})\right]^{z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

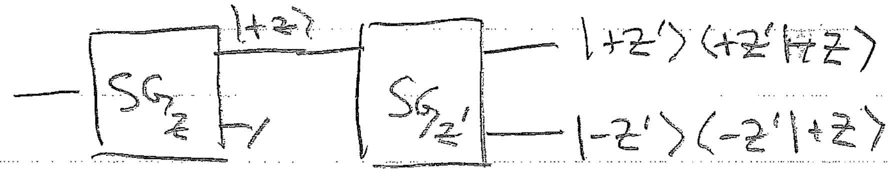


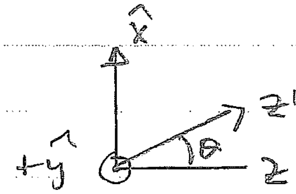
corresponding to $|\pm x\rangle = \hat{R}_S(\tfrac{\pi}{2}\hat{y})\,|\pm z\rangle$ or

$$\left(|+x\rangle, |-x\rangle\right) = \left(|+z\rangle, |-z\rangle\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

note overall phase for $|-x\rangle$ from usual definition

<u>Example</u>





$|\pm z'\rangle = \hat{R}^S(\theta\hat{y})\,|\pm z\rangle$

$\langle +z'|+z\rangle = \langle +z|\hat{R}^{S\dagger}(\theta\hat{y})\,|+z\rangle = \cos\frac{\theta}{2}$

$\langle -z'|+z\rangle = \langle -z|\hat{R}^{S\dagger}(\theta\hat{y})\,|+z\rangle = -\sin\theta/2$

probabilities to measure $\pm\hbar/2$ along z' are $\cos^2\theta/2$, $\sin^2\theta/2$ as we found in recitation problem

Spin-operator transform as Euclidean vector. $\langle \hat{S}_i \rangle$ are components of spin angular momentum which must transform as Euclidean vector.

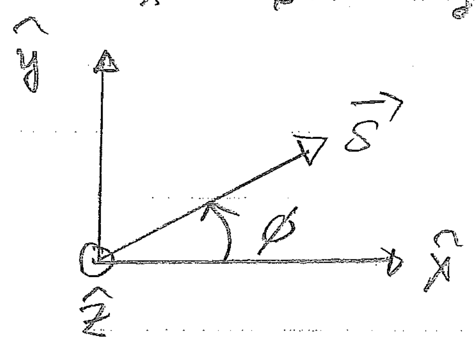
<u>Proof</u>: (see for example Sakurai <u>Quantum mechanics</u>)

Expand exponentials and use $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$

$$\hat{R}^{s\dagger}(\phi\hat{z})\, \hat{S}_x\, \hat{R}^s(\phi\hat{z})$$

$$= \exp\left(\frac{i\hat{S}_z\phi}{\hbar}\right) \hat{S}_x \exp\left(-\frac{i\hat{S}_z\phi}{\hbar}\right)$$

$$= \hat{S}_x \cos\phi - \hat{S}_y \sin\phi$$



Therefore, expectation values of spin transform as components of a Euclidean vector. The measured spin is an angular momentum which is a Euclidean vector.

Note on change of basis

$$|b_i\rangle = \sum_j |a_j\rangle\langle a_j|b_i\rangle$$

$$|b_i\rangle = \hat{S}\,|a_i\rangle \qquad \hat{S} \text{ unitary operator}$$

$$|b_i\rangle = \sum_j |a_j\rangle\langle a_j|\hat{S}|a_i\rangle = \sum |a_j\rangle [\hat{S}]^a_{ji}$$

summed row index

For example, $|\pm\hat{y}\rangle = \hat{S}\,|\pm\hat{z}\rangle$

We know

$$|\pm y\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle \pm i\,|-z\rangle\right)$$

I write this way to get $[\hat{S}]^z_{ij}$ immediately

$$\left(|+y\rangle, |-y\rangle\right) = \left(|+z\rangle, |-z\rangle\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Or you can say

$$[\hat{S}]^z_{ij} = \begin{pmatrix} \langle +z|\hat{S}|+z\rangle, & \langle +z|\hat{S}|-z\rangle \\ \langle -z|\hat{S}|+\hat{z}\rangle, & \langle -z|\hat{S}|-z\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle +z|+y\rangle, & \langle +z|-y\rangle \\ \langle -z|+y\rangle, & \langle -z|-y\rangle \end{pmatrix}$$

$$\langle +z|+y\rangle = \langle +z|\frac{1}{\sqrt{2}}\left(|+z\rangle + i\,|-z\rangle\right) = \frac{1}{\sqrt{2}}, \text{ etc}$$

$$[S]^z_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{which is much more work!}$$