Physics 491
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Lecture # 12: Groups & Representations

Continuous (Lie) Group:

$$g(\vec{\theta}) = \exp\left(-i \sum_{i=1}^{N_g} \theta_i \hat{T}_i\right)$$

$\hat{T}_i$ are hermitian generators, N_g is the dimension (dim) of group.

Examples in physics

group	generator	invariance ⇒ conservation law
rotations SU(2)	$\vec{J}$	angular momentum
translations in space	$\vec{p}$	momentum
translations in time	$\hat{H}$ Hamiltonian	energy
local U(1) gauge	e	charge

$$\psi \to e^{ie\Lambda(\vec{r},t)/\hbar c}\,\psi(\vec{r},t)$$

Λ arbitrary function of space & time

A <u>representation</u> is a mapping of $\hat{T}_i$ to explicit operators acting on an <u>n-dim vector space</u> (i.e. $n \times n$ matrices)

<u>n is the dim of the representation</u> (rep)

Almost all group properties are determined by algebra of generators

$$\left[\hat{T}_i, \hat{T}_j\right] = i\sum_k f_{ijk} \hat{T}_k$$

f_{ijk} = structure constants

<u>Lowest dim rep</u> is called the <u>fundamental</u> or <u>defining</u> rep.

<u>SU(2)</u>: 2×2, det = 1 unitary matrices
corresponding generators are <u>traceless</u>, <u>hermitian</u> matrices $\hat{\sigma}_i/2$

<u>generators</u>: $\frac{\sigma_x}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\frac{\sigma_y}{2} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\frac{\sigma_z}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\sum_k \varepsilon_{ijk} \frac{\sigma_k}{2}$$

ε_{ijk} are structure constants of SU(2).

group dim = 3 (in general, $SU(N)$ dim = N^2-1)

Defining rep group elements are

spin rotations

$$g(\vec{\theta}) = \exp\left[-i \sum_{i=1}^{3} \theta_i \frac{\sigma_i}{2}\right]$$

$$= \exp\left[-i\, \vec{\theta}\cdot\vec{\sigma}/2\right]$$

since $\vec{\sigma}_i$ transform as Euclidean vector

Reps of $SU(2)$ (irreducible ≡ no invariant subgroups)

dim	angular momentum eigenvalue	state or particle
$\underset{\sim}{1}$	0	scalar Higgs
$\underset{\sim}{2}$	$-\frac{1}{2}, \frac{1}{2}$	spin-$\frac{1}{2}$ e, p
$\underset{\sim}{3}$	$-1, 0, 1$	spin-1 γ, W, Z
$\underset{\sim}{4}$	$-3/2, -1/2, \frac{1}{2}, 3/2$	spin-3/2 Gravitino?
$\underset{\sim}{5}$	$-2, -1, 0, 1, 2$	spin-2 graviton?

SU(2) representation theory

start with $\left[\frac{\hat{J}_i}{\hbar}, \frac{\hat{J}_j}{\hbar}\right] = i\,\varepsilon_{ijk}\,\frac{\hat{J}_k}{\hbar}$

Basis states defined by maximal set of commuting operators. For SU(2)

one generator $\hat{J}_3/\hbar$

Casimir operator $\hat{J}^2/\hbar^2$

$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$ commutes with all generators,

$$[\hat{J}^2, \hat{J}_i] = 0$$

label states by (so far unknown) eigenvalues

$$\hat{J}^2 |\lambda, m\rangle = \hbar^2 \lambda |\lambda, m\rangle$$
$$\hat{J}_3 |\lambda, m\rangle = \hbar\, m |\lambda, m\rangle$$

First, $m^2 \le \lambda$ since $\hat{J}_i$ are hermitian,

$$\langle \hat{J}_i^2 \rangle \geq 0$$

Hermitian eigenvalues are real,
(Hermitian)^2 eigenvalues non-negative.

so $\langle \lambda, m | (\hat{J}_1^2 + \hat{J}_2^2) | \lambda, m \rangle \geq 0$

or $\langle \lambda, m | (\hat{J}^2 - \hat{J}_3^2) | \lambda, m \rangle \geq 0$

$$\lambda - m^2 \geq 0$$

Raising and lowering ("ladder") operators

$$\hat{J}_\pm \equiv \hat{J}_1 \pm i\hat{J}_2$$

$$\hat{J}_+^\dagger = \hat{J}_-$$

Useful property:

$$[\hat{J}_3, \hat{J}_\pm] = [\hat{J}_3, \hat{J}_1] \pm i[\hat{J}_3, \hat{J}_2]$$

$$= i\hbar \hat{J}_2 \pm i(-i\hbar \hat{J}_1)$$

$$= \pm\hbar(\hat{J}_1 \pm i\hat{J}_2) = \pm\hbar\hat{J}_\pm$$

$$\boxed{[\hat{J}_3, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm}$$

$\hat{J}_\pm$ raise (lower) eigenvalue m:

$$\hat{J}_3(\hat{J}_\pm | \lambda, m\rangle) = (\hat{J}_\pm \hat{J}_3 + \underbrace{[\hat{J}_3, \hat{J}_\pm]}_{\pm\hbar\hat{J}_\pm}) | \lambda, m\rangle$$

$$= \hbar(m \pm 1)(\hat{J}_\pm | \lambda, m\rangle)$$

So $\quad \hat{J}_\pm | \lambda, m\rangle = C_\pm | \lambda, m \pm 1\rangle$

$C_\pm$ determined by normalization.

$$\hat{J}_+\hat{J}_- = (\hat{J}_1 + i\hat{J}_2)(\hat{J}_1 - i\hat{J}_2)$$

$$= \hat{J}^2 - \hat{J}_3^2 + \hbar\hat{J}_3$$

$$\hat{J}_-|\lambda, m\rangle = C_-|\lambda, m-1\rangle$$

$$|C_-|^2 = \langle\lambda, m|\hat{J}_+\hat{J}_-|\lambda, m\rangle$$

$$= \langle\lambda, m|(\hat{J}^2 - \hat{J}_3^2 + \hbar J_3)|\lambda, m\rangle$$

$$(\lambda - m^2 + m)\hbar^2$$

$$C_- = \hbar\sqrt{\lambda - m(m-1)}$$ similarly

$$C_+ = \hbar\sqrt{\lambda - m(m+1)}$$

there must be a min and max m.
let max $m = j$, min $m = -j$. then

$\hat{J}_+|\lambda, j\rangle = 0$ and therefore

$$\hat{J}_-\hat{J}_+|\lambda, j\rangle = (\hat{J}^2 - \hat{J}_3^2 - \hbar\hat{J}_3)|\lambda, j\rangle$$

compare to $\hat{J}_+\hat{J}_-$

$$= \hbar^2(\lambda - j(j+1))|\lambda, j\rangle$$

$$\boxed{\lambda = j(j+1)}$$

customary to label basis by $|j, m\rangle$.

$$\boxed{\hat{J}_\pm |j,m\rangle = \hbar\left[j(j+1) - m(m\pm 1)\right]^{1/2} |j, m\pm 1\rangle}$$

dim of rep $n = 2j+1$

example $\hat{J}_x$ in vector rep 3 $(j=1)$

$$\hat{J}_+ |1,-1\rangle = \hbar\sqrt{2-(-1)(-1+1)}\,|1,0\rangle$$

$$= \hbar\sqrt{2}\,|1,0\rangle$$

$$\hat{J}_+ |1,0\rangle = \hbar\sqrt{2}\,|1,1\rangle$$

$$\hat{J}_+ = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

similarly $\hat{J}_- = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$\hat{J}_x = \frac{1}{2}\left([J_+] + [J_-]\right) = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$