Physics 491
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Lecture 15: multiparticle States

State describing spin of multiple electrons — build up as direct product:

$$|+z;-z\rangle \equiv |+z\rangle_1 \otimes |-z\rangle_2 \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

a 4 dim. vector space.

example: Hyperfine splitting of hydrogen

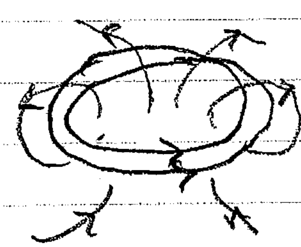
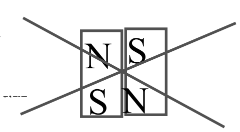
$$\hat{H}_{hf} = \frac{2A}{\hbar^2} \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

where A is an energy. Magnetic moment parallel (spin anti-parallel) states have lower energy. Field of magnetic dipole,

$$\vec{B} = \frac{1}{r^3}\left[3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}\right] + \frac{8\pi}{3}\vec{\mu}\,\delta^3(\vec{r})$$

only this piece contributes to ground state

Intuitively: lower energy state corresponds to overlapping current loops with // currents:

think  not 

states $|+,+\rangle, |-,-\rangle$ have energy $A/2$

linear combination of $|+,-\rangle, |-,+\rangle$ are the ground state. How to find the ground state?

Method A: Diagonalize $\hat{H}$. Trick to write down $\hat{H}$:

$$\boxed{2\,\hat{\vec{S}}^e\cdot\hat{\vec{S}}^p = \hat{S}^e_+\hat{S}^p_- + \hat{S}^e_-\hat{S}^p_+ + 2\hat{S}^e_z\hat{S}^p_z}$$

$$S^e_+|-\rangle_e = \hbar\left[\tfrac{1}{2}(\tfrac{1}{2}+1)-(-\tfrac{1}{2})(-\tfrac{1}{2}+1)\right]^{1/2}|+\rangle_e = \hbar|+\rangle_e$$

$$S^e_-|+\rangle_e = \hbar\left[\tfrac{3}{4}-(\tfrac{1}{2})(\tfrac{1}{2}-1)\right]^{1/2} = \hbar|-\rangle_e$$

so $S^e_+S^p_-|-,+\rangle = \hbar^2|+,-\rangle$ leading to

$$[\hat{H}] = \frac{A}{2}\begin{array}{c|cccc|c} & ++ & +- & -+ & -- & \\ \hline & 1 & 0 & 0 & 0 & ++ \\ & 0 & -1 & 2 & 0 & +- \\ & 0 & 2 & -1 & 0 & -+ \\ & 0 & 0 & 0 & 1 & -- \end{array}$$

states $|+,+\rangle, |-,-\rangle$ are energy eigenstates with eigenvalue $+\frac{A}{2}$.

States $|2\rangle \equiv |+,-\rangle$ and $|3\rangle \equiv |-,+\rangle$ *mix*.

$$\frac{A}{2}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} = E\begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix}$$

$$|\psi\rangle = \psi_2 |+;-\rangle + \psi_3 |-;+\rangle$$

Symmetric

$$|1,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|+;-\rangle + |-;+\rangle \right), \quad E_{sym} = \frac{A}{2}$$

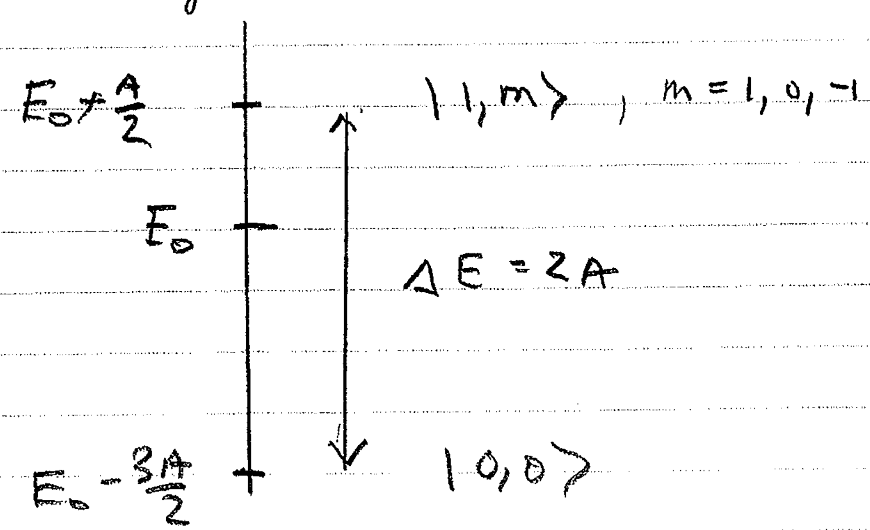
antisymmetric

$$|0,0\rangle \equiv \frac{1}{\sqrt{2}}\left(|+;-\rangle - |-;+\rangle \right); \quad E_{anti} = -\frac{3A}{2}$$

where new notation refers to the total spin eigenvalues $|j,m\rangle = |1,m\rangle$ triplet
$|j,m\rangle = |0,0\rangle$ singlet

We will see that $|1,1\rangle = |+;+\rangle$, $|1,-1\rangle = |-;-\rangle$

level diagram



Note on matrix $\hat{H} = \frac{2A}{\hbar^2}\vec{\hat{S}}^e \cdot \vec{\hat{S}}^p$

It is 4x4 because vector space is 4 dim:

$$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$$

using a simplified notation $|++\rangle \equiv |\tfrac{1}{2}, +\tfrac{1}{2}\rangle_e |\tfrac{1}{2}, +\tfrac{1}{2}\rangle_p$
etc. Rather than writing

$$2\vec{S}^e \cdot \vec{S}^p = \hat{S}_+^e \hat{S}_-^p + \hat{S}_-^e \hat{S}_+^p + 2\hat{S}_z^e \hat{S}_z^p$$

We can use $\vec{S}^e \cdot \vec{S}^p = \hat{S}_x^e \hat{S}_x^p + \hat{S}_y^e \hat{S}_y^p + \hat{S}_z^e \hat{S}_z^p$

$$\hat{S}_x^e |+\rangle_e \rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \frac{\hbar}{2}|-\rangle_e$$

Similarly $\hat{S}_x^e |-\rangle_e = \frac{\hbar}{2}|+\rangle_e$

and $\hat{S}_y |\pm\rangle_e = \pm i \frac{\hbar}{2}|\mp\rangle_e$

then

$$\hat{S}_x^e \hat{S}_x^p |\pm\pm\rangle = \frac{\hbar^2}{4}|\mp\mp\rangle \;;\; \hat{S}_y^e \hat{S}_y^p |\pm\pm\rangle = -\frac{\hbar^2}{4}|\mp\mp\rangle$$

$$\hat{S}_x^e \hat{S}_x^p |\pm\mp\rangle = \frac{\hbar^2}{4}|\mp\pm\rangle \;;\; S_y^e S_y^p |\pm\mp\rangle = \frac{\hbar^2}{4}|\mp\pm\rangle$$

Giving

$$\begin{array}{cccc} & x & y & z \\ \hat{H} = & \frac{A}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & + \frac{A}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} & + \frac{A}{2}\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix} \end{array}$$

$$= \frac{A}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Angular Momentum Method

$\hat{H}$ is rotationally invariant.

$$\hat{H} = \frac{2A}{\hbar^2} \hat{S}^e \cdot \hat{S}^p = \frac{A}{\hbar^2}\left(\hat{S}^2 - \hat{S}^{e\,2} - \hat{S}_p^2\right)$$

where $\vec{S} \equiv \vec{S}^e + \vec{S}^p$ total angular momentum
(note: ground state of hydrogen has zero orbital angular momentum.)

since $\left[\hat{S}^{e\,2}, \hat{S}^2\right] = 0 = \left[\hat{S}_p^2, \hat{S}^2\right]$, $\left[\hat{H}, \hat{S}^2\right] = 0$

Eigenstates of $\hat{H}$ are eigenstates of $\hat{S}^2$.

We already know that $|+;+\rangle$ is an eigenstate of $\hat{H}$:

$$\hat{H}|+;+\rangle = \frac{A}{\hbar^2}\left(\hat{S}^e_+\hat{S}^p_- + \hat{S}^e_-\hat{S}^p_+ + 2\hat{S}^e_z\hat{S}^p_z\right)|+;+\rangle$$

$$= \frac{A}{2}|+;+\rangle$$

What is $\hat{S}^2|+;+\rangle$?

$$\hat{S}^{e\,2}|+;+\rangle = \hbar^2 \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)|+;+\rangle = \hbar^2\tfrac{3}{4}|+;+\rangle$$

$$\frac{A}{\hbar^2}\left(\hat{S}^2 - \hat{S}^{e\,2} - \hat{S}_p^2\right)|+;+\rangle = \hat{H}|+;+\rangle = \frac{A}{2}|+;+\rangle$$

$$\hat{S}^2|+;+\rangle = \hbar^2\left(\tfrac{1}{2} + \tfrac{3}{4} + \tfrac{3}{4}\right)|+;+\rangle = 2\hbar^2|+;+\rangle$$

$$= \hbar^2\left(1(1+1)\right)|+;+\rangle$$

corresponding to $j=1$, $m=1$.

We can find other state with <u>same j=1</u> by applying the lowering operator:

$$\hat{S}_- = \hat{S}_-^e + \hat{S}_-^p$$

$$\hat{S}_-^e |+\rangle_e = \hbar^2 \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\ |-\rangle_e = \hbar\, |-\rangle_e$$

$$\hat{S}_- |+;+\rangle = (\hat{S}_-^e + \hat{S}_-^p)\,|+;+\rangle$$

$$= \hbar\left(|-;+\rangle + |+;-\rangle\right)$$

$$= \sqrt{2}\hbar\Big[\underbrace{\tfrac{1}{\sqrt{2}}\left(|-;+\rangle + |+;-\rangle\right)}_{\text{normalized}}\Big] = \sqrt{2}\hbar\,|1,0\rangle$$

<u>notice</u> $\hat{S}_- |1,1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\ |1,0\rangle = \sqrt{2}\hbar\,|1,0\rangle$

Applying $\hat{S}_-$ again,

$$\hat{S}_- |1,0\rangle = (\hat{S}_-^e + \hat{S}_-^p)\frac{1}{\sqrt{2}}\left(|-;+\rangle + |+;-\rangle\right)$$

$$= \sqrt{2}\hbar\,|-;-\rangle = \sqrt{2}\hbar\,|1,-1\rangle$$

we can check the consistency of $\hat{S}_z = \hat{S}_z^e + \hat{S}_z^p$

$$\hat{S}_z |+;+\rangle = \frac{\hbar}{2}|+;+\rangle + \frac{\hbar}{2}|+;+\rangle = \hbar\,|+;+\rangle$$

and $\hat{S}_z |-;-\rangle = -\hbar\,|-;-\rangle$

and $\hat{S}_z |1,0\rangle = 0$

What we have done is constructed the $\underset{\sim}{3}$ of SU(2) from a product:

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} + \underset{\sim}{1} \qquad \text{multiplicities}$$

or in terms of j: " $\frac{1}{2} \times \frac{1}{2}$ " = " $1 + 0$ "

The states are products, with angular momentum eigenvalues j that have been added:

$$j_{TOT} = \tfrac{1}{2} + \tfrac{1}{2} = 1$$
$$j_{TOT} = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

in general, $j_{TOT} = j_1 + j_2,\ j_1 + j_2 - 1,\ \dots,\ |j_1 - j_2|$

symmetry ensures that:

① $\hat{H}\,|1,m\rangle = \frac{A}{2}\,|1,m\rangle$ states of same j have same energy (are degenerate)

② states have same symmetry under $e \leftrightarrow p$

What is singlet? Must be orthogonal to $|1,0\rangle$:

$$|0,0\rangle = \frac{1}{\sqrt{2}}\big(\,|-;+\rangle - |+;-\rangle\,\big)$$

What is $\hat{\vec{S}}^2|0,0\rangle$ and $\hat{H}|0,0\rangle$?

Write $\hat{H} = \frac{A}{\hbar^2}\left(\hat{S}^e_+\hat{S}^p_- + \hat{S}^e_-\hat{S}^p_+ + 2\hat{S}^e_z\hat{S}^p_z\right)$

note that $\hat{S}^e_+\hat{S}^p_-|+;-\rangle = 0 = \hat{S}^e_-\hat{S}^p_+|-;+\rangle$

and $\hat{S}^e_-\hat{S}^p_+|+;-\rangle = \hbar^2|-;+\rangle$

$$\hat{S}^e_z S^p_z|+;-\rangle = -\frac{\hbar^2}{4}|+;-\rangle$$

so $2\hat{\vec{S}}^e\cdot\vec{S}^p|0,0\rangle =$

$$\frac{1}{\sqrt{2}}\Big[\hat{S}^e_-\hat{S}^p_+|+;-\rangle - \hat{S}^e_+\hat{S}^p_-|-;+\rangle + 2\hat{S}^e_z\hat{S}^p_z|+;-\rangle - 2\hat{S}^e_z\hat{S}^p_z|-;+\rangle\Big]$$

$$= -\frac{3}{2}\frac{\hbar^2}{\sqrt{2}}\Big[|+;-\rangle - |-;+\rangle\Big] = \underline{\underline{-\frac{3}{2}}}\hbar^2|0,0\rangle$$

$$\boxed{\hat{H}|0,0\rangle = -\frac{3}{2}A|0,0\rangle}$$

Check $2\hat{\vec{S}}^e\cdot\hat{\vec{S}}^p|0,0\rangle = -\frac{3}{2}\hbar^2|0,0\rangle = \left(\hat{S}^2 - \hat{S}_e^2 - \hat{S}_p^2\right)|0,0\rangle$

with $\hat{S}_e^2|0,0\rangle = \hat{S}_e^2\frac{1}{\sqrt{2}}\left(|+;-\rangle - |-;+\rangle\right)$

$$= \hbar^2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)\frac{1}{\sqrt{2}}\left(|+;-\rangle - |-;+\rangle\right) = \frac{3}{4}|0,0\rangle$$

So $\hat{S}^2|0,0\rangle = \left(\hat{S}_e^2 + \hat{S}_p^2 + 2\hat{\vec{S}}^e\cdot\hat{\vec{S}}^p\right)|0,0\rangle$

$$= \left(\frac{3}{4} + \frac{3}{4} - \frac{3}{2}\right)|0,0\rangle = 0.$$

as expected.

Comments :

Under rotations, the $\underset{\sim}{3}$ $|1,1\rangle, |1,0\rangle, |1,-1\rangle$ transform as the $\underset{\sim}{3}$ of SU(2). This means that any rotation will be a linear combination of these three and will not mix with the singlet $|0,0\rangle$. The combinations $\underset{\sim}{3}$, $\underset{\sim}{1}$ are said to be irreducible.

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$$

reducible product state $\rightarrow$ $\Big\}$ irreducible linear combination of product state

Applying the general rule ; eigenvalues of total $\vec{J} = \vec{j}_1 + \vec{j}_2$:

$$j = j_1 + j_2, \; j_1 + j_2 - 1, \; \ldots, \; |j_1 - j_2|$$

We can determine the multiplicities of the irreducible combinations !

$$\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} + \underset{\sim}{2} \quad \left(\text{"} 1 \times \tfrac{1}{2} \text{"} = \text{"} \tfrac{3}{2} + \tfrac{1}{2} \text{"} \right)$$

$$\underset{\sim}{3} \times \underset{\sim}{3} = \underset{\sim}{5} \oplus \underset{\sim}{3} + \underset{\sim}{1} \quad \left(\text{"} 1 \times 1 \text{"} = \text{"} 2 + 1 + 0 \text{"} \right)$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \times \underset{\sim}{2} = \underset{\sim}{2} \otimes \left(\underset{\sim}{3} + \underset{\sim}{1} \right)$$

$$= \underset{\sim}{4} + \underset{\sim}{2} + \underset{\sim}{2}$$

differ by symmetry of (1,2,3) combinations.

$$\text{"}\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2}\text{"} = \text{"}\tfrac{3}{2} + \tfrac{1}{2} + \tfrac{1}{2}\text{"}$$

proton and neutron have the same nuclear interaction. There is therefore a symmetry called "isospin". This symmetry is carried by the constituent quarks u,d.

the baryons are observed in multiplets of "isospin" $\frac{1}{2}$ and $3/2$!

$$\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} uud \\ udd \end{pmatrix}$$

where $q_u = \frac{2}{3}$, $q_d = -\frac{1}{3}$

$$\begin{pmatrix} \Delta^{++} \\ \Delta^{+} \\ \Delta^{0} \\ \Delta^{-} \end{pmatrix} = \begin{pmatrix} uuu \\ uud \\ udd \\ ddd \end{pmatrix}$$

And the corresponding anti-particles.
the Δ^{--} does not exist!

Murray Gell-Mann's "Eight Fold Way" 1961 based on the group SU(3) with $3^2-1 = 8$ generators.

Clebsch-Gordon coefficients

Rule in SU(2) for irreducible representation decomposition

$$|j_1 - j_2| \le J \le j_1 + j_2$$

in integer steps

$j_1 \times j_2$	J m	J m	...
m_1 m_2	Coefficients		
m_1 m_2			

read as $"-\frac{8}{15}" \rightarrow -\sqrt{\frac{8}{15}}$

In j notation; In multiplicity notation:

example $\underset{\sim}{1} \otimes \underset{\sim}{\tfrac{1}{2}} = \underset{\sim}{3/2} \oplus \underset{\sim}{1/2}$; $\underset{\sim}{3} \times \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2}$

same number of states

read table to get:

$$\underbrace{|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle}_{\text{sum to } \frac{1}{2}} = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},+\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

or read in other direction

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,\underbrace{|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle}_{\text{sum to } 1/2} + \sqrt{\tfrac{2}{3}}\,\underbrace{|1,0\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle}_{\text{sum to}}$$