Phys 491
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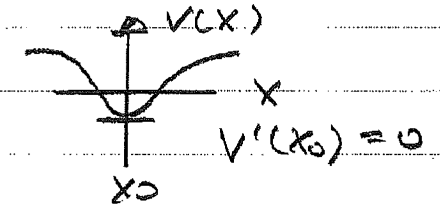
Lecture 18: Harmonic Oscillator

(Almost) Any stable equilibrium has an approximately quadratic potential energy.

$$V(x) = V(x_0) + (x-x_0)V'(x_0) + \frac{1}{2}(x-x_0)^2 V''(x_0) + \cdots$$



so long as $V''(x_0) \neq 0$, $\quad V(x) \approx \frac{1}{2}k(x-x_0)^2$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \quad , \omega = \sqrt{\frac{k}{m}}$$

Solution: for energy eigenstates from time independent equation

① analytic (diff. eq.) straightforward

② algebraic (operator) elegant

<u>dimensionless variables:</u>

$$\hat{q} \equiv \hat{p}/\sqrt{m\hbar\omega} \; ; \; \hat{y} \equiv \hat{x}\sqrt{\frac{m\omega}{\hbar}}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{1}{2}\hbar\omega(\hat{q}^2 + \hat{y}^2)$$

$$\frac{2E}{\hbar\omega} \equiv \mathcal{E}$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$[\hat{y}, \hat{q}] = i \Rightarrow \hat{q} \to -i\frac{\partial}{\partial y}$$

Analytic Solution - Hermite Polynomials

$$-\frac{d^2\psi}{dy^2} + y^2\psi = \varepsilon\psi$$

$$\frac{d^2\psi}{dy^2} + (\varepsilon - y^2)\,\psi = 0$$

① asymptotic behavior as $|y| \to \infty$

$$\frac{d^2\psi}{dy^2} \simeq y^2\psi$$

Approximate
normalizable solution $\psi \xrightarrow[|y|\to\infty]{} Ae^{-y^2/2}$

let $\psi(y) = h(y)\, e^{-y^2/2}$ and get

$$\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (\varepsilon - 1)h = 0$$

power series solution : $h(y) = \sum_{k=0}^{\infty} a_k y^k$

recursion relation

$$\frac{a_{k+2}}{a_k} = \frac{2k+1-\varepsilon}{(k+2)(k+1)} \xrightarrow[k\to\infty]{} \frac{2}{k}$$

Expansion for large k is same as

$$e^{y^2} = \sum_{n=0}^{\infty} \frac{y^{2n}}{n!} = \sum_{\substack{k=0 \\ k\ \text{even}}}^{\infty} \underbrace{\frac{1}{\left(\frac{k}{2}\right)!}}_{\equiv b_k} y^k$$

$$\frac{b_{k+2}}{b_k} = \frac{(k/2)!}{\left(\frac{k}{2}+1\right)!} = \frac{1}{\frac{k}{2}+1} \xrightarrow[k\to\infty]{} \frac{2}{k}$$

thus $h(y) \xrightarrow[|y|\to\infty]{} e^{y^2}$

and $\psi(y) \xrightarrow[|y|\to\infty]{} e^{y^2} e^{-y^2/2} = e^{+y^2/2} \to \infty$

unless series terminates. So

$$2n+1-\varepsilon = 0 \quad \text{for some } k \equiv n$$

$$\varepsilon = 2n+1$$

$$\boxed{E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)}$$ quantized energy

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

properly normalized

Hermite polynomials $H_n(y)$

$H_0 = 1$ even

$H_1 = 2y$ odd

$H_2 = 4y^2 - 2$ "

$H_3 = 8y^3 - 12y$

$H_4 = 16y^4 - 48y^2 + 12$

generating function:

$$H_N(y) = (-1)^N e^{y^2} \frac{\partial^N}{\partial y^N} e^{-y^2}$$

series expansion:

$$e^{-z^2+2zy} = \sum_{N=0}^{\infty} \frac{z^N}{N!} H_N(y)$$

Algebraic Solution

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{q}^2 + \hat{y}^2\right)$$

$$[\hat{y}, \hat{q}] = i$$

$$\hat{q} \xrightarrow[y\text{-basis}]{} -i\frac{\partial}{\partial y}$$

$$\hat{H} \rightarrow \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right)$$

Dirac: $\hat{H}$ factorize

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{y} + i\hat{q})$$

$$[\hat{y}, \hat{q}] = i \quad \Rightarrow \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{a}^\dagger\hat{a} = \frac{1}{2}(\hat{y} - i\hat{q})(\hat{y} + i\hat{q}) = \frac{1}{2}\left\{\hat{y}^2 + \hat{q}^2 + i[\hat{y}, \hat{q}]\right\}$$

$$= \frac{1}{2}(\hat{y}^2 + \hat{q}^2) - \frac{1}{2}$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}$$

$\hat{a}^\dagger\hat{a} \equiv \hat{N}$ "number" operator, up to constant equal to $\hat{H}$.

Define number basis:

$$\hat{N}|n\rangle = n|n\rangle$$

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{N}+\tfrac{1}{2}\right)|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle$$

So far, all we know is that n is a real, dimensionless number.

Consider $[\hat{N},\hat{a}] = \hat{a}^{\dagger}\hat{a}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{a}$

$$= (\hat{a}\hat{a}^{\dagger}-1)\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{a} = -\hat{a}$$

and, taking hermitian conjugate

$$[\hat{a}^{\dagger},\hat{N}] = -\hat{a}^{\dagger}$$

$$[\hat{N},\hat{a}^{\dagger}] = +\hat{a}^{\dagger}$$

Compare to algebra of $\hat{J}_z$, $\hat{J}_{\pm}$

$$[\hat{J}_z, \hat{J}_{\pm}] = \pm\hbar J_{\pm}$$

$$\left.\begin{aligned}\hat{a}^{\dagger} &\sim \hat{J}_{+}\\ \hat{a} &\sim J_{-}\end{aligned}\right\}$$ $\hat{a}^{\dagger}$, $\hat{a}$ are raising & lowering operators for states $|n\rangle$

$$\hat{N}\left(\hat{a}^{\dagger}|n\rangle\right) = \left(\hat{a}^{\dagger}\hat{N} + \hat{a}^{\dagger}\right)|n\rangle = (n+1)\left(\hat{a}^{\dagger}|n\rangle\right)$$

so $\hat{a}^{\dagger}|n\rangle = C_{+}|n+1\rangle$

Similarly $\hat{a}|N\rangle = C_-|N-1\rangle$

physically, there must be a ground state:

$$\hat{a}\,|N_{min}\rangle = 0$$

eigenvalue $\hat{N}\,|N_{min}\rangle = \hat{a}^+\hat{a}\,|N_{min}\rangle = 0$

so $|N_{min}\rangle = |0\rangle$ and n is an integer

$$\hat{H}\,|N\rangle = \hbar\omega\left(N+\tfrac{1}{2}\right)|N\rangle$$

$$E_N = \hbar\omega\left(n+\tfrac{1}{2}\right) \quad , N = 0, 1, 2$$

normalization: start with $\langle 0|0\rangle = 1$

$$\hat{a}\,|N\rangle = C_-\,|N-1\rangle$$
$$\langle N|\,\hat{a}^+ = C_-^*\,\langle N-1|$$

$$\langle n|\,\hat{a}^+\hat{a}\,|n\rangle = n\,\langle N|N\rangle = |C_-|^2\,\langle N-1|N-1\rangle$$

if $\langle N|N\rangle = 1$, $|C_-| = \sqrt{N}$

and $|C_+| = \sqrt{N+1}$

$$\boxed{|N+1\rangle = \frac{\hat{a}^+}{(N+1)^{1/2}}\,|N\rangle}$$

$$|1\rangle = \hat{a}^+|0\rangle$$

$$|2\rangle = \frac{\hat{a}^+}{\sqrt{2}}|1\rangle = \frac{(\hat{a}^+)^2}{\sqrt{2}}|0\rangle$$

$$|3\rangle = \frac{\hat{a}^+}{\sqrt{3}}|2\rangle = \frac{(\hat{a}^+)^3}{\sqrt{3!}}|0\rangle$$

$$\boxed{|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle}$$

In the $\hat{y}$-rep, $\hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{y} - i\hat{g}) = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$

⊛ $$\langle y|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n\left(y - \frac{d}{dy}\right)^n \langle y|0\rangle$$

$\langle y|0\rangle$ is the solution to a simple 1^{st} order diff. eq.

$$0 = \langle \hat{y}|\hat{a}|0\rangle = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)\langle y|0\rangle$$

$$\frac{d}{dy}\langle y|0\rangle = -y\langle y|0\rangle$$

Exact normalized solution

$$\boxed{\langle y|0\rangle = \frac{1}{\pi^{1/4}} e^{-y^2/2}}$$

From ⊛ We get another formula for generating Hermite polynomials:

$$\boxed{\left(y - \frac{d}{dy}\right)^n e^{-y^2/2} = H_n(y)\, e^{-y^2/2}}$$

Uncertainty Product for H.O. states

$$\Delta x = \sqrt{\frac{\hbar}{m\omega}}\,\Delta y \quad , \quad \Delta p = \sqrt{\hbar m\omega}\,\Delta q$$

$$\Delta x\,\Delta p = \hbar\,\Delta y\,\Delta q$$

$$\langle n|\hat{y}|n\rangle = \langle n|\tfrac{1}{\sqrt{2}}(\hat{a}^{+}+\hat{a})|n\rangle = 0$$

since $\langle n|m\rangle = \delta_{nm}$

similarly $\langle n|\hat{q}|n\rangle = 0$

$$\hat{y}^2 = \frac{1}{2}(a^{+}+a)(a^{+}+a) = \frac{1}{2}(a^{+2}+a^2+\underbrace{a^{+}a+aa^{+}}_{2\hat{N}+1})$$

$$= \frac{1}{2}(\hat{a}^{+2}+\hat{a}^2) + \hat{N} + \frac{1}{2}$$

$$\langle n|\hat{y}^2|n\rangle = n + \frac{1}{2}$$

similarly, $\hat{q}^2 = -\frac{1}{2}(\hat{a}^{+2}+\hat{a}^2) + \hat{N} + \frac{1}{2}$

$$\Delta y\,\Delta q = n + \frac{1}{2} \implies \Delta x\,\Delta p = \hbar\left(n+\frac{1}{2}\right)$$

ground state (Gaussian) has minimum uncertainty product $\hbar/2$.

<u>time evolution of $\langle x \rangle$, $\langle p \rangle$</u>

$$\frac{d\langle \hat{x} \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{x}] \rangle$$

with $\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^+ + \hat{a}\right)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^+ - \hat{a}\right)$$

$$[\hat{H}, \hat{x}] = \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left[a^+a, a^+ + a\right]$$

$$= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left(\underbrace{[a^+a, a^+]}_{\hat{a}^+} + \underbrace{[a^+a, a]}_{-\hat{a}}\right)$$

$$= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^+ - \hat{a}\right) = \frac{\hbar}{im}\hat{p}$$

similarly $[\hat{H}, \hat{p}] = i\hbar m\omega^2\hat{x}$ therefore

$$\boxed{\begin{aligned} \frac{d}{dt}\langle \hat{x} \rangle &= \frac{1}{m}\langle \hat{p} \rangle \\ \frac{d}{dt}\langle \hat{p} \rangle &= \frac{i}{\hbar}\langle [\hat{H}, \hat{p}] \rangle = -m\omega^2\langle \hat{x} \rangle \end{aligned}}$$