Physics 491

Nov 28, 2016 <u>Lecture 19: Coherent States</u>

Time evolution of arbitrary superposition of harmonic oscillator energy eigenstates $|n\rangle$!

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle$$

$$\left.\begin{aligned}\frac{d}{dt}\langle\hat{x}\rangle &= \frac{1}{m}\langle\hat{p}\rangle \\ \frac{d}{dt}\langle\hat{p}\rangle &= -m\omega^2\langle\hat{x}\rangle\end{aligned}\right\} \frac{d^2}{dt^2}\langle\hat{x}\rangle = -\omega^2\langle\hat{x}\rangle$$

$$\langle\hat{x}(t)\rangle = \langle\hat{x}(0)\rangle\cos\omega t + \frac{\langle\hat{p}(0)\rangle}{m\omega}\sin\omega t$$

however, $\langle n|\hat{x}|n\rangle = 0$ and $\langle n|\hat{p}|n\rangle = 0$

We can find superposition states that have non-zero $\langle\hat{x}(0)\rangle$, $\langle\hat{p}(0)\rangle$ and have a minimum uncertainty product (Gaussians in space) $\Delta x \Delta p_x = \hbar/2$.

These are displaced ground state wave functions called <u>coherent states</u>.

Schrödinger 1926; Roy Glauber Physical Review <u>131</u> 2766 (1963) space, momentum displacement operators

$$\hat{T}(y_0) = e^{-iy_0\hat{q}} \quad ; \quad \hat{W}(q_0) = e^{+iq_0\hat{y}}$$

$$\langle y | \hat{W}(q_0)\hat{T}(y_0)|0\rangle \equiv \langle y|z\rangle =$$

$$e^{-iq_0 y}\langle y - y_0|0\rangle = \frac{e^{-iq_0 y}}{\pi^{1/4}} e^{-(y-y_0)^2/2}$$

Expand $|z\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|z\rangle$

$$|z\rangle = e^{iq_0\hat{y}} e^{-iy_0\hat{p}}|0\rangle$$

use $e^{\hat{A}}e^{\hat{B}} = e^{\frac{1}{2}C} e^{\hat{A}+\hat{B}}$ where $C = [\hat{A},\hat{B}]$ a number

$$[iq_0\hat{y}, -iy_0\hat{q}] = y_0 q_0 [\hat{y},\hat{q}] = iy_0 q_0$$

$$|z\rangle = e^{iy_0 q_0/2} \exp(iq_0\hat{y} - iy_0\hat{q})|0\rangle$$

$$\hat{y} = \frac{1}{\sqrt{2}}(\hat{a}^+ + \hat{a}) \quad ; \quad \hat{q} = \frac{i}{\sqrt{2}}(\hat{a}^+ - \hat{a})$$

so $$iq_0\hat{y} - iy_0\hat{q} = \hat{a}^+ \underbrace{\left(\frac{y_0 + iq_0}{\sqrt{2}}\right)}_{\equiv z} - \hat{a}\underbrace{\left(\frac{y_0 - iq_0}{\sqrt{2}}\right)}_{= z^*}$$

$$|z\rangle = e^{iy_0 q_0/2} \exp(\hat{a}^+ z - \hat{a} z^*)|0\rangle$$

$$|z\rangle = e^{i\frac{\gamma_0 \xi_0}{2}} e^{-\frac{1}{2}[\hat{a}^+ z, -\hat{a} z^*]} e^{\hat{a}^+ z} e^{-\hat{a} z^*} |0\rangle$$

$$e^{-\hat{a} z^*} |0\rangle = |0\rangle \text{ since } \hat{a}|0\rangle = 0$$

$$[\hat{a}^+ z, -\hat{a} z^*] = |z|^2 [\hat{a}, \hat{a}^+] = |z|^2$$

we can ignore pure phase $e^{i\gamma_0 \xi_0/2}$

$$|z\rangle = e^{-|z|^2/2} \underbrace{e^{\hat{a}^+ z}}_{\sum_{n=0}^{\infty} \frac{(\hat{a}^+ z)^n}{n!}} |0\rangle$$

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{(\hat{a}^+ z)^n}{n!} |0\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

so amplitude $\langle n|z\rangle = e^{-|z|^2/2} \frac{z^n}{\sqrt{n!}}$

$$P_n = |\langle n|z\rangle|^2 = \frac{|z|^{2n}}{n!} e^{-|z|^2}$$

<u>Poisson with mean $|z|^2$</u>

$$\langle E\rangle = \sum_{n=0}^{\infty} E_n |\langle n|z\rangle|^2 = \hbar\omega \sum_{n=0}^{\infty} \left(n + \tfrac{1}{2}\right) \frac{|z|^{2n}}{n!} e^{-|z|^2}$$

$$= \hbar\omega e^{-|z|^2} \Bigg(\underbrace{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n-1)!}}_{|z|^2 e^{|z|^2}} + \frac{1}{2} \underbrace{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}}_{e^{|z|^2}} \Bigg)$$

giving $\langle E \rangle = \hbar\omega\left(|z|^2 + \frac{1}{2}\right)$

$$|z|^2 = \frac{1}{2}\left(y_0^2 + \xi_0^2\right)$$

<u>Time dependence</u>:

$$\hat{U}(t)\,|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, e^{-i(n+\frac{1}{2})\omega t}\, |n\rangle$$

$$= e^{-i\omega t/2}\, e^{-|z|^2} \sum_n \frac{\left(z e^{-i\omega t}\right)^n}{\sqrt{n!}}\, |n\rangle$$

define $z(t) \equiv z e^{-i\omega t}$ and ignore overall phase:

$$\hat{U}(t)\,|z\rangle = e^{-|z|^2} \sum_n \frac{\left(z(t)\right)^n}{\sqrt{n!}}\, |n\rangle = |z(t)\rangle$$

then $\langle \hat{y}(t) \rangle = \langle z(t)|\, \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) |z(t)\rangle$

$$\hat{a}\,|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, \underbrace{\hat{a}\,|n\rangle}_{\sqrt{n}\,|n-1\rangle}$$

$$= z\, e^{|z|^2/2} \sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n-1)!}}\, |n-1\rangle = z\,|z\rangle$$

$\boxed{\hat{a}\,|z\rangle = z\,|z\rangle}$ often, this is definition of $|z\rangle$.

and $\hat{a}^\dagger\,|z\rangle = z^*\,|z\rangle$

$$\langle \hat{y}(t) \rangle = \frac{1}{\sqrt{2}}\left(z^*(t) + z(t)\right)$$

finally, $z(t) = \frac{1}{\sqrt{2}}(y_0 + i g_0)\underbrace{e^{-i\omega t}}_{\cos\omega t - i\sin\omega t}$

giving $\boxed{\langle \hat{y}(t) \rangle = y_0 \cos\omega t + g_0 \sin\omega t}$

and $\langle \hat{g}(t) \rangle = -y_0 \sin\omega t + g_0 \cos\omega t$

y_0 is initial ground state displacement,
g_0 is initial momentum.

<u>Classical limit</u>

$$\langle E \rangle = \hbar\omega \left(|z|^2 + \frac{1}{2} \right)$$

and you can show

$$\Delta E = \hbar\omega |z|$$

then $\frac{\Delta E}{\langle E \rangle} \xrightarrow[\text{large}|z|]{} \frac{1}{|z|}$

<u>example</u>: classical oscillator

$m = 10g \quad \ell = 0.1m \quad g = 10\,m/s^2$

$x_0 = 10\,cm \quad p_0 = 0$

$T = 2\pi\sqrt{\frac{\ell}{g}} = 0.63\,s \quad \underline{\omega = \frac{2\pi}{T} = 10\,rad/s}$

$\hbar \cong 10^{-34}\,J.s$

$|z| = \frac{y_0}{\sqrt{2}} = x_0\sqrt{\frac{m\omega}{2\hbar}} = x_0\sqrt{5}\,10^{+16}\,m^{-1} = \underline{2\times10^{+15}}$

$\frac{\Delta E}{E} = \frac{1}{|z|} = \frac{1}{2}\times10^{-15}$

Quantum Optics

Quantized EM field is collection of harmonic oscillators, one for each frequency ω –

$$a_\omega^+ |0\rangle = |1\rangle_\omega \quad \text{single photon} \quad E = \hbar\omega$$

Consider light of single frequency and drop ω subscript. Define phase

$$z = |z| e^{i\phi}$$

phase operator, (Dirac)

$$\hat{a} = e^{i\hat{\phi}} \sqrt{\hat{N}}$$

$$\hat{a}^+ = \sqrt{\hat{N}}\, e^{-i\hat{\phi}}$$

$$\hat{a}^+ \hat{a} = \sqrt{\hat{N}}\, e^{-i\hat{\phi}} e^{i\hat{\phi}} \sqrt{\hat{N}} = \hat{N}$$

Consider $\left[\hat{N}, e^{i\hat{\phi}}\right] = \left[\hat{N}, \hat{a}\frac{1}{\sqrt{\hat{N}}}\right]$

$$= \hat{N}\hat{a}\frac{1}{\sqrt{\hat{N}}} - \hat{a}\sqrt{\hat{N}} = \left[\hat{N}, \hat{a}\right]\frac{1}{\sqrt{\hat{N}}}$$

$$= \Big(\underbrace{\hat{a}^+\hat{a}}_{aa^+ - 1}\hat{a} - \hat{a}\hat{a}^+\hat{a}\Big)\frac{1}{\sqrt{\hat{N}}} \qquad \left\{ \begin{array}{l} [a^+, a] = -1 \\ a^+a = aa^+ - 1 \end{array} \right.$$

$$= -\hat{a}\frac{1}{\sqrt{\hat{N}}} = -e^{i\phi}$$

So $\left[\hat{N}, 1 - i\hat{\phi} + \frac{i^2\hat{\phi}^2}{2!} + \dots\right] = -\left(1 - i\hat{\phi} + \frac{i^2\hat{\phi}^2}{2!} + \dots\right)$

$$\boxed{\text{So } \left[\hat{N}, \hat{\phi}\right] = i}$$

we therefore have

$$\Delta n \, \Delta \phi \geq \frac{1}{2}$$

Single photons do not have defined phase

Coherent states are Poisson

$$|\langle n|z\rangle|^2 = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!}$$

Poisson $\langle n\rangle = (\Delta n)^2 = |z|^2$

$$\Delta \phi = \frac{1}{2} \frac{1}{\sqrt{\langle n\rangle}}$$

Classical EM wave can have definite phase with very large mean number of photons