Phys 491 Fall 2021

Lecture #20: Propagator

Given $\psi(x_0, t_0)$ what is $\psi(x_1, t_1)$ $t_1 > t_0$?

$$|\psi(t_1)\rangle = \hat{U}(t_1 - t_0)|\psi(t_0)\rangle$$

we assume $\hat{H}$ independent of time

$$\hat{U}(t_1 - t_0) = \exp(-i\hat{H}(t_1 - t_0)/\hbar)$$

$$\langle x_1|\psi(t_1)\rangle = \psi(x_1 t_1) =$$

$$\int dx_0 \underbrace{\langle x_1|\exp(-i\hat{H}(t_1-t_0)/\hbar)|x_0\rangle}_{\text{propagator or Green's function } \langle x_1,t_1|x_0,t_0\rangle} \overbrace{\langle x_0|\psi(t_0)\rangle}^{\psi(x_0,t_0)}$$

propagator or Green's function $\langle x_1 t_1|x_0,t_0\rangle$

Free partical propagator: $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\langle x, t_1|x_0, t_0\rangle_{free} = \langle x_1| e^{-i\frac{\hat{p}^2}{2m\hbar}(t_1 - t_0)}|x_0\rangle$$

eigenvalue

$$= \int dp \, \langle x_1|p\rangle e^{-i\frac{p^2}{2m\hbar}(t_1-t_0)} \langle p|x_0\rangle$$

plane wave $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar}$

exponential argument is

$$-i\frac{p^2}{2m\hbar}(t_1 - t_0) + \frac{ix_1 p}{\hbar} - i\frac{x_0 p}{\hbar}$$

$$= -\frac{i}{\hbar}\left[\frac{p^2}{2m}(t_1 - t_0) - p(x_1 - x_0)\right]$$

A Gaussian (quadratic in exponential)

$$\langle x_1 t_1 | x_0 t_0 \rangle_{free} = \int \frac{dp}{2\pi\hbar} \exp -\frac{i}{\hbar}\left[\frac{p^2}{2m}(t_1 - t_0) - p(x_1 - x_0)\right]$$

by completing the square and integrating

$$\langle x_1 t_1 | x_0 t_0 \rangle_{free} = \sqrt{\frac{m}{2\pi i \hbar (t_1 - t_0)}} \exp\left\{\frac{i m (x_1 - x_0)^2}{2\hbar (t_1 - t_0)}\right\}$$

$$\{\ \} = \frac{i}{\hbar} \frac{m}{2} V^2 T$$

$$V = \frac{x_1 - x_0}{t_1 - t_0} \qquad T = t_1 - t_0$$

this is the <u>classical action</u>

In Lagrangian formulation for single particle

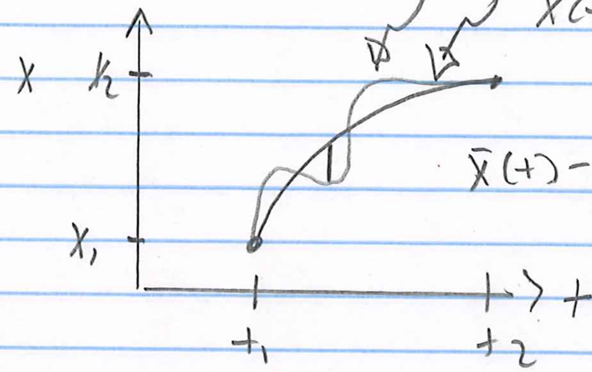
$$\mathcal{L}(\dot{x}, x) = T - V = \frac{m\dot{x}^2}{2} - V(x)$$

action S is functional of path $x(t)$:

$$S[x(t)] = \int_{t_0}^{t_1} \mathcal{L}(\dot{x}, x)\, dt$$

varied path $x(t)$

$\bar{x}(t)$ classical solution



$\bar{x}(t) - x(t) \equiv \delta x(t)$ variation

$$S[\bar{x}+\delta x] = \int_{t_0}^{t_1} \mathcal{L}(\dot{\bar{x}} + \delta\dot{x}, \bar{x}+\delta x)\, dt$$

Taylor expand

$$= \int_{t_0}^{t_1} \left[\mathcal{L}(\dot{\bar{x}}, \bar{x}) + \delta\dot{x}\, \frac{\partial \mathcal{L}}{\partial \dot{x}} + \delta x\, \frac{\partial \mathcal{L}}{\partial x} \right.$$

integrate by parts

$$\int \frac{d}{dt}(\delta x) \left.\frac{\partial \mathcal{L}}{\partial \dot{x}}\right| dt = \delta x \left.\frac{\partial \mathcal{L}}{\partial \dot{x}}\right|_{t_0}^{t_1} - \int \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\, \delta x\, \delta t$$

Variation vanishes at endpoints: 0 variation $\delta x(t_1) = \delta x(t_0) = 0$

$$S[\bar{x}+\delta x] = \underbrace{\int_{t_0}^{t_1} \mathcal{L}\, dt}_{S[\bar{x}]} + \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x\, \delta t$$

Classical path is the extremum

$$\delta S \equiv S[\bar{x}+\delta x] - S[\bar{x}] = 0 = \int \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x\, \delta t$$

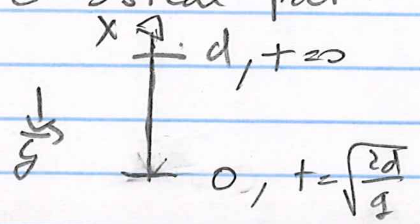
Since δx is arbitrary, classical path satisfies Lagrange eq.

what is $S[\bar{x}]$ numerically? take $V(x) = mgx$

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - mgx \qquad x(t) = -\frac{1}{2}gt^2 + d$$

classical path $\quad (x_0, t_0) = (d, 0)$

$$(x_1, t_1) = \left(0, \sqrt{\frac{2d}{g}}\right)$$



$$S[\bar{x}] = \int_0^{\sqrt{\frac{2d}{g}}} \left(\frac{m\dot{x}^2}{2} - mgx\right) dt$$

$$= \int_0^{\sqrt{\frac{2d}{g}}} \left[\frac{m}{2}(-gt)^2 - mg\left(d - \frac{gt^2}{2}\right)\right] dt$$

$$= \int_0^{t_1} (mgt^2 - mgd)\, dt = \frac{1}{3}mg^2 t_1^3 - mgdt_1$$

$$= \sqrt{\frac{2d}{g}}\left[\frac{1}{3}mg^2\left(\frac{2d}{g}\right) - mgd\right] = -\frac{1}{3}md\sqrt{2dg}$$

dim $\;kg\, m^2/s = J\cdot s$

classical object $\quad m = 1\,g \quad d = 10\,cm \quad g = 10\, m/s^2$

$$(10^{-3}\,kg)(0.1\,m)\sqrt{2(0.1m)10\, m/s^2} = 10^{-4}\sqrt{2}\; J\cdot s$$

$$\frac{S[\bar{x}]}{\hbar} = \frac{10^{-4}\sqrt{2}}{3}\;\frac{J\cdot s}{10^{-34}\,J\cdot s} \simeq 10^{+30}$$

However, consider electron moving with constant acceleration $a = \frac{e\mathcal{E}}{m_e}$.

$$S[\bar{x}] = -\frac{1}{3} m d \sqrt{2da}$$

kinetic energy $KE(eV) = e \underbrace{\mathcal{E}}_{\text{volts / meter}} \overbrace{d}^{\text{distance}}$

$$m d \sqrt{2d\left(\frac{e\mathcal{E}}{m}\right)} = d\sqrt{2m \cdot KE}$$

$$\frac{S}{\hbar} = -\frac{1}{3}\left(\frac{d}{\hbar c}\right)\sqrt{2(mc^2)KE} = -\frac{1}{3}\frac{d}{\underbrace{\hbar c}_{200\ eV\cdot nm}}\sqrt{10^6 eV\ KE[eV]}$$

$$= -\frac{1}{3}\frac{d[nm]\,10^3\sqrt{KE[eV]}}{200}$$

$$= -\frac{10}{6}\, d[nm]\sqrt{KE[eV]}$$

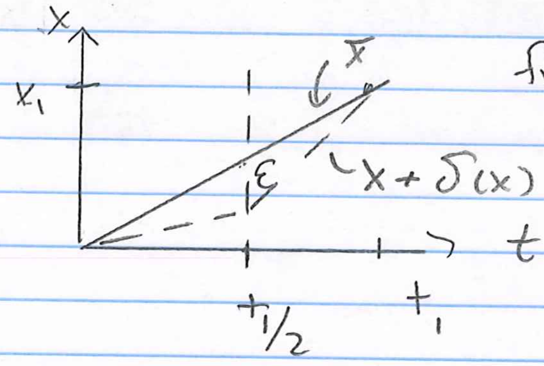
so QM object can have

$$\frac{S}{\hbar} \sim 1$$

Feynman's Conjecture:

$$\langle x_1, t_1 | x_0, t_0 \rangle \sim \sum_{\text{all paths}} e^{iS/\hbar}$$

The idea: $S[\bar{x} + \delta x]$ varies quadratically with deviation from extremal path.



free particle: $v = \dfrac{x_1}{t_1}$

$$S[\bar{x}] = \frac{m}{2}\left(\frac{x_1}{t_1}\right)^2 t_1$$

$$S[\bar{x} + \delta x] = \left(\frac{t_1}{2}\right)\frac{m}{2}\left[\left(\frac{\frac{x_1}{2} - \varepsilon}{t_1/2}\right)^2 + \left(\frac{\frac{x_1}{2} + \varepsilon}{t_1/2}\right)^2\right]$$

$$= S[\bar{x}] + \frac{t_1}{2}\left(\frac{m}{2}\right)\frac{8\varepsilon^2}{t_1^2} = S[\bar{x}]\left(1 + 4\left(\frac{\varepsilon}{x_1}\right)^2\right)$$

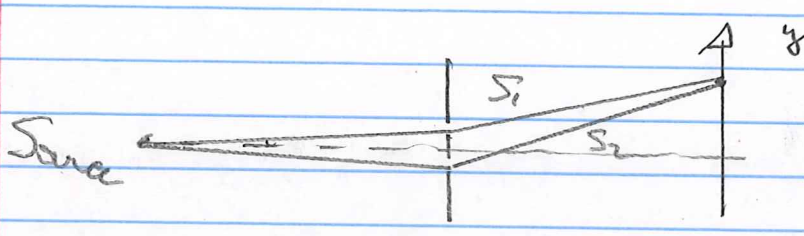
$$\delta S = S[\bar{x}]\, 4\left(\frac{\varepsilon}{x_1}\right)^2$$

Sum of complex exponential cos, sin will sum to zero (wash out) when $\delta S \gg \hbar$. Classical particle follows classical path because $\delta S \gg \hbar$ except when $\delta S = 0$ on classical path. Q.M. trajectories can sum coherently when $\delta S \sim \hbar$.

We saw for accelerated electron

$$S \sim d[nm]\sqrt{KE(eV)}$$

So low KE electrons can diffract in double slit experiment.



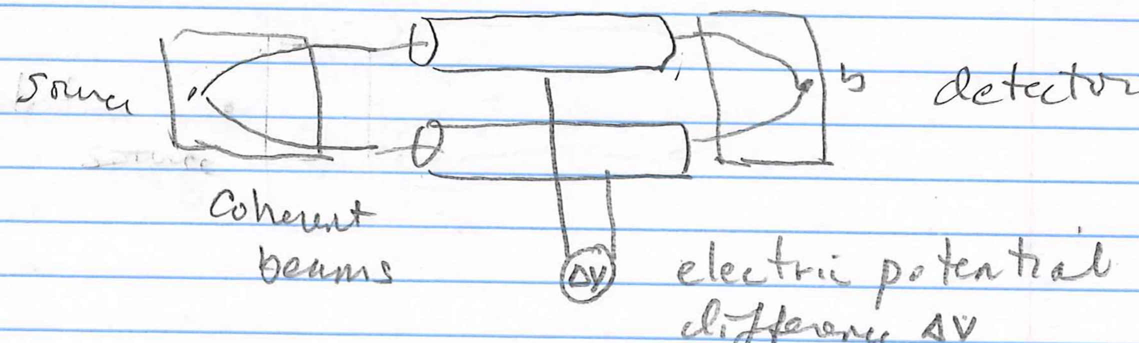
$$A \sim \psi\left(e^{iS_1/\hbar} + e^{iS_2/\hbar}\right)$$
$$= \psi e^{iS_1/\hbar}\left(1+e^{i\Delta S/\hbar}\right)$$

single electron takes both paths and interferes

$$P = |A|^2 = \frac{1}{2}\left[1 + \cos\left(\frac{\Delta S}{\hbar}\right)\right]$$

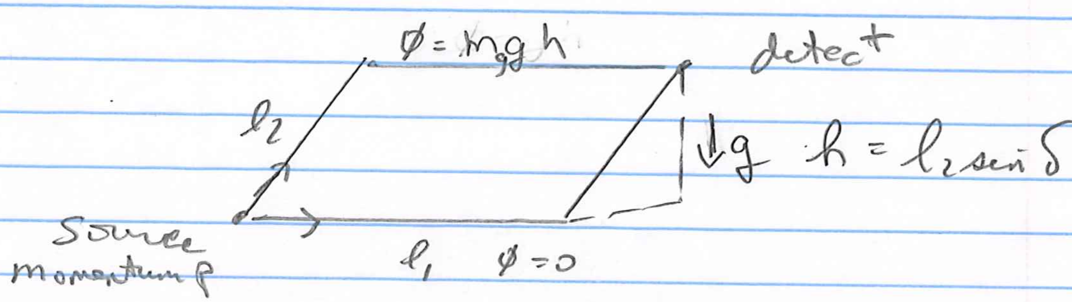
Some suprising quantum interference effects:

<u>Drift tubes</u>



$\vec{E} = 0$ inside tubes. no classical force
but potential difference gives: $\Delta S = (e\Delta V)$ time
fringe field effects can be arbitrarily small

Tilted plane in gravitational field w/ neutrons



m_g = gravitational mass

m_i = inertial mass

assuming neutron loses negligible momentum going up in gravitational field

$$\Delta S = \phi \cdot \text{time} = \left(m_g g\, l_2 \sin\delta\right) \frac{l_1 m_i}{p}$$

$$\Delta S \propto m_i m_g$$

Can observe interference and test principle of equivalence $m_i = m_g$.

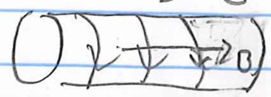
(see text)

Aharnov-Bohm

$$\mathcal{L}_{Em} = -e\phi + \frac{e}{c}\vec{A}\cdot\vec{v} \qquad q = -e$$

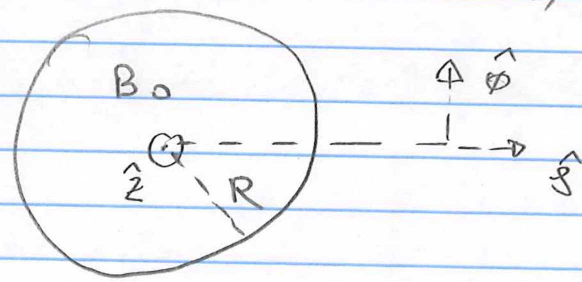
Consider long solinoid of radius R

$\vec{B} = 0$



$\rightarrow \hat{z}$ inside $\vec{B} = B_0\hat{z}$

outside, $\vec{B} = 0$



cylindrical coordinate

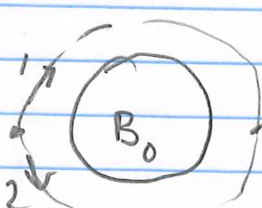
$$\rho > R \qquad \vec{A}_{out} = \frac{B_0 R^2}{2\rho}\hat{\phi}$$

$$\rho < R \qquad \vec{A}_{in} = \frac{1}{2}\left(\frac{B_0\rho}{2}\right)\hat{\phi}$$

$$B_z = \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi)\right) = \begin{cases} B_0 & \rho < R \\ 0 & \rho > R \end{cases}$$

Two path interference:

electron source



detector

$$A = \psi\left(1 + e^{i\Delta S/\hbar}\right)$$

$$\Delta S = \frac{e}{c}\oint \vec{A}\cdot d\vec{s} = \frac{e}{c}\Phi_B \qquad \Phi_B = B_0 \pi R^2$$

closed path 1 minus 2 Flux

Interference is

$$P = \frac{1}{2}\left[1 + \cos\left(\frac{e\Phi_B}{\hbar c}\right)\right]$$

P varies sinusoidally with field strength B_0.

There is no magnetic field where electrons move.

Φ_B is <u>gauge invariant</u>