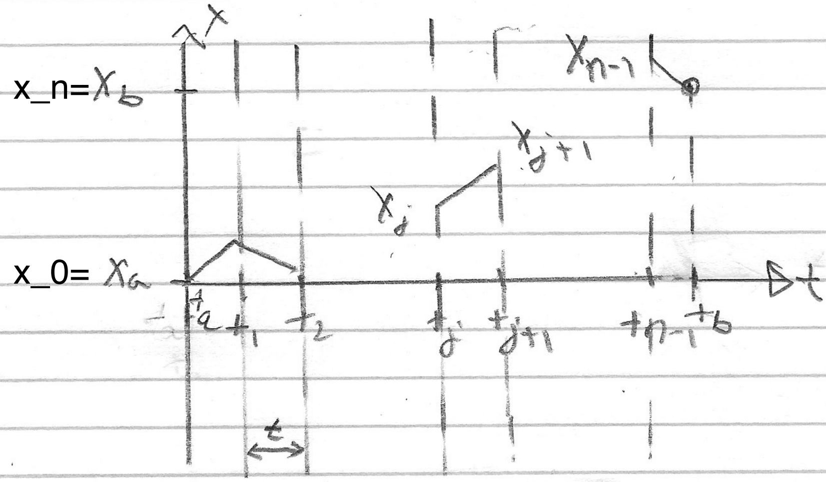
Phys 491 Fall 2021

Lecture #21 Feynman Path Integral

Proving Feynman conjecture. How to sum over paths?



$\frac{t_b - t_a}{n} = t$

n links

n-1 intermediate times

define

$t_a \equiv t_0$

$t_b \equiv t_n = t_a + nt$

propagator $\langle x_b, t_b | x_a, t_a\rangle = \langle x_b | \hat{U}(t_b - t_a) | x_a \rangle$

$\hat{U}(t_b - t_a) = \exp\left[-\frac{i}{\hbar}\hat{H}(t_b - t_a)\right] = \left(\hat{U}(t)\right)^n$

then insert unit operator $\hat{I} = \int dx_j\, |x_j\rangle\langle x_j|$ n-1 times one for each intermediate time

$$\langle x_b, t_b | x_a, t_a \rangle = \int dx_1 \cdots \int dx_{n-1} \prod_{j=0}^{n-1} \langle x_{j+1} | \hat{U}(t) | x_j \rangle$$

note all space integrals go from -infinity to +infinity

for small t $\hat{U}(t) \simeq 1 - \frac{i}{\hbar} t H(\hat{x}, \hat{p})$

$$= 1 - \frac{i}{\hbar} t \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)$$

$V(\hat{x})|x_j\rangle = V(x_j)|x_j\rangle$

we have to also use $\hat{p}|p_j\rangle = p_j|p_j\rangle$

Insert $\hat{I} = \int dp_j \, |p_j\rangle\langle p_j|$ j=1,N one for each link

all momentum integrals go from -infinity to +infinity

and use $\hat{p}\,|p_j\rangle = p_j\,|p_j\rangle$

$$\int dp_j \, \langle x_{j+1}| \left[1 - \frac{it}{\hbar}\left(\frac{\hat{p}^2}{2m} + V(x_j)\right)\right] |p_j\rangle\langle p_j|x_j\rangle$$

$$= \int dp_j \, \langle x_{j+1}|p_j\rangle\langle p_j|x_j\rangle \left[1 - \frac{it}{\hbar}\left(\frac{p_j^2}{2m} + V(x_j)\right)\right]$$

define $E_j = \frac{p_j^2}{2m} + V(x_j)$

and approximate $\left[1 - \frac{it}{\hbar}E_j\right] \simeq e^{-iE_j t/\hbar}$

use explicit plane waves

$$\langle x_{j+1}|p_j\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(i p_j x_{j+1}/\hbar\right)$$

add arguments of the exponentials

$$i\frac{p_j x_{j+1}}{\hbar} - i\frac{p_j x_j}{\hbar} - i\frac{E_j t}{\hbar} = \frac{it}{\hbar}\left[p_j \frac{(x_{j+1} - x_j)}{t} - E_j\right]$$

notice that $\left[p_j\left(\frac{x_{j+1} - x_j}{t}\right) - E_j\right] \sim p_j \dot{x}_j - E_j = \mathcal{L}$

Recall Legendre transformation:

$$\mathcal{L}(\dot{x}, x) + \mathcal{H}(p, x) = p\dot{x}$$

function of $\dot{x}, x$ function of p, x

putting in n insertion of $\int dp_j \, |p_j\rangle\langle p_j|$

$$\langle x_b, t_b | x_a, t_a \rangle \cong \left(\prod_{j=1}^{n-1} \int dx_j \right) \left(\prod_{j=1}^{n} \int \frac{dp_j}{2\pi\hbar} \right)$$

$$\times \prod_{j=1}^{n} \exp \frac{i\tau}{\hbar} \left[p_j \frac{x_{j+1} - x_j}{\tau} - E_j \right]$$

$$= (\;)(\;) \exp \frac{i\tau}{\hbar} \sum_{j=1}^{n} \left[\frac{p_j (x_{j+1} - x_j)}{\tau} - E_j \right]$$

in limit $n \to \infty$ $\tau \to 0$ we get phase space propagator

$$\exp \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left(p\dot{x} - E(p, x) \right) \equiv \exp \frac{i}{\hbar} \underbrace{S(t_a, t_b)}_{\text{action}}$$

functional measures $\mathcal{D}[x(t)]$, $\mathcal{D}[p(t)]$ are $n \to \infty$ limits

$$\langle x_b, t_b | x_a, t_a \rangle = \int_{x_a}^{x_b} \mathcal{D}[x(t)] \int \mathcal{D}[p(t)] \, e^{\frac{i}{\hbar} S(t_a, t_b)}$$

A very formal expression

Feynman path integral (FPI)

for $\quad H = \frac{p^2}{2m} + V(x)$ we can explicitly do p_j integrals by completing the square.

$$\int \frac{dp_j}{2\pi\hbar} \exp \frac{i}{\hbar} \left\{ p_j (x_{j+1} - x_j) - t\frac{p_j^2}{2m} - tV(x_j) \right\}$$

$$= e^{-itV(x_j)/\hbar} \sqrt{\frac{m}{2\pi i \hbar t}} \exp \frac{it}{\hbar} \left[\frac{m}{2} \left(\frac{x_{j+1} - x_j}{t} \right)^2 \right]$$

again summing exponentials

$$\lim_{n \to \infty} \exp \frac{i}{\hbar} \sum t \left[\frac{m}{2} \left(\frac{x_{j+1} - x_j}{t} \right)^2 - V(x_j) \right]$$

$$= \exp \frac{i}{\hbar} \int_{t_a}^{t_b} dt \underbrace{\left[\frac{m}{2} \dot{x}^2 - V(x) \right]}_{\mathcal{L}(\dot{x}, x)}$$

$$= \exp \frac{i}{\hbar} S[x(t)]$$

giving FPI

$$\langle x_b, t_b | x_a, t_a \rangle = \int_{x_a}^{x_b} \mathcal{D}[x(t)] \, e^{\frac{i}{\hbar} S[x(t)]}$$

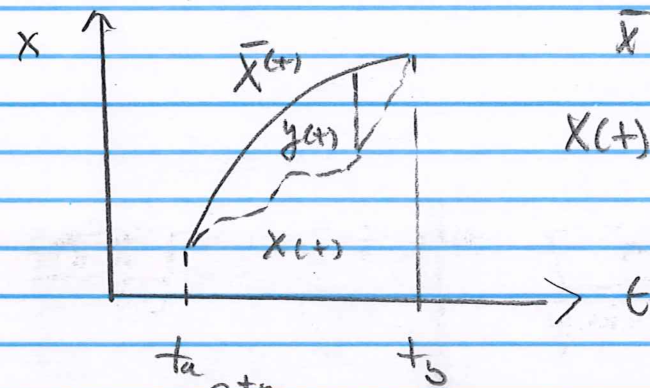
$$S[x(t)] = \int_{t_a}^{t_b} dt \, \mathcal{L}(\dot{x}, x)$$ classical action along path $x(t)$.

FPI sum over paths

Evaluating FPI

for $\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + a + bx + cx^2 + d\dot{x} + ex\dot{x}$

a, b, c, d, e possibly time dependant



$\bar{x}$ classical path

$x(t) = \bar{x}(t) + y(t)$

$$S[x] = S[\bar{x}+y] = \int_{t_a}^{t_b} \mathcal{L}(\bar{x}+y, \dot{\bar{x}}+\dot{y})$$

$$\mathcal{L}(x,\dot{x}) \cong \mathcal{L}(\bar{x},\dot{\bar{x}}) + y\left.\frac{\partial\mathcal{L}}{\partial x}\right|_{\bar{x},\dot{\bar{x}}} + \dot{y}\left.\frac{\partial\mathcal{L}}{\partial\dot{x}}\right|_{\bar{x},\dot{\bar{x}}}$$

$$+ \frac{1}{2}y^2\left.\frac{\partial^2\mathcal{L}}{\partial x^2}\right|_{\bar{x},\dot{\bar{x}}} + y\dot{y}\left.\frac{\partial^2\mathcal{L}}{\partial x\,\partial\dot{x}}\right|_{\bar{x},\dot{\bar{x}}} + \frac{1}{2}\dot{y}^2\left.\frac{\partial^2\mathcal{L}}{\partial\dot{x}^2}\right|_{\bar{x},\dot{\bar{x}}}$$

terms linear in $y, \dot{y}$ vanish due to extremal condition:

$$\int dt\left[\dot{y}\left.\frac{\partial\mathcal{L}}{\partial\dot{x}}\right| + y\left.\frac{\partial\mathcal{L}}{\partial x}\right|\right) = \underbrace{y\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big|_{t_a}^{t_b}}_{=0 \text{ at } t_a, t_b} +$$

$$\int_{t_a}^{t_b} dt\, y\left[\left.\frac{\partial\mathcal{L}}{\partial x}\right|_{\bar{x},\dot{\bar{x}}} - \frac{d}{dt}\left.\frac{\partial\mathcal{L}}{\partial\dot{x}}\right|_{\bar{x},\dot{\bar{x}}}\right] = 0$$

so $S[x] = S[\bar{x}] + \int_{t_a}^{t_b} dt\left[\frac{1}{2}m\dot{y}^2 + ey\dot{y} + cy^2\right]$

$$\langle x_b, t_b | x_a, t_a \rangle = A(t_b - t_a)\, e^{i S[\bar{x}]/\hbar}$$

$S[\bar{x}]$ is action on classical path, and

$$A(t_b - t_a) = \int_0^0 \mathcal{D}[y(t)] \exp \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left(\tfrac{1}{2} m \dot{y}^2 + e y \dot{y} + c y^2 \right)$$

Comments

① Schrödinger equation can be derived from path integral (see Feynman + Hibbs)

② $$\langle x_b | \psi(t_b) \rangle = A(t_b - t_a) \int_{-\infty}^{\infty} dx_a\, e^{i S[\bar{x}]/\hbar} \langle x_a | \psi(t_a) \rangle$$

③ $A(t)$ can be evaluated from simple differential equation.

L.S. Schulman, Techniques and Applications of Path Integrals

$$A(t) = \sqrt{\frac{m}{2\pi i \hbar f(t)}}$$ where $$m \frac{\partial^2 f}{\partial t^2} + C f = 0$$

boundary conditions $f(0) = 0$, $\left.\frac{\partial f}{\partial t}\right|_{t=0} = 1$

example: Free partical $(c = 0)$

$$\frac{\partial^2 f}{\partial t^2} = 0 \Rightarrow f = t$$

$$A(t) = \sqrt{\frac{m}{2\pi i \hbar t}}$$ our previous result!

Harmonic oscillator:

$$S[\bar{x}] = \frac{m\omega}{2\sin\omega t}\left[\cos\omega t\,(x_b^2 + x_a^2) - 2x_a x_b\right]$$

$$m\frac{\partial^2 f}{\partial t^2} + \frac{1}{2}m\omega^2 f = 0 \qquad \begin{matrix} f(0)=0 \\ f'(0)=1 \end{matrix}$$

$$f = \frac{\sin\omega t}{\omega}$$

$$A(t) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}$$

Feynman-Kac analytic continuation

to imaginary time $t \to -i\tau$ $\quad e^{-iEt/\hbar} \to e^{-E\tau/\hbar}$

Start with propagator expanded in complete set of energy eigenstates $\hat{H}|n\rangle = E_n|n\rangle$.

$$\langle x_b|\hat{U}(\tau)|x_a\rangle = \sum \langle x_b|e^{-\tau\hat{H}/\hbar}|n\rangle\langle n|x_a\rangle$$

$$= \sum \langle x_b|n\rangle\langle n|x_a\rangle e^{-\tau E_n/\hbar}$$

$$\xrightarrow[\tau\to\infty]{} \langle x_b|0\rangle\langle 0|x_a\rangle e^{-\tau E_0/\hbar} \quad \text{ground state}$$

analytic continuation of Harmonic Oscillator

$$\sin(-i\omega\tau) = \frac{1}{2i}\left(e^{+\omega\tau} - e^{-\omega\tau}\right) = \frac{1}{i}\sinh(\omega\tau)$$

$$\cos(-i\omega\tau) = \cosh(\omega\tau)$$

$$S[\bar{x}] = \frac{i m\omega}{2\sinh(\omega\tau)}\left[\cosh\omega\tau\,(x_b^2+x_a^2) - 2x_a x_b\right]$$

in limit $\tau \to \infty$ $\quad \sinh(\omega\tau) \to \frac{1}{2}e^{\omega\tau} \quad \cosh(\omega\tau) \to \frac{1}{2}e^{\omega\tau}$

$$\frac{iS[\bar{x}]}{\hbar} \xrightarrow[\tau\to\infty]{} \frac{-m\omega}{2\hbar} \, 2e^{-\omega\tau}\left[\frac{e^{\omega\tau}}{2}\left(x_a^2 + x_b^2\right)\right]$$

$$= \frac{-m\omega}{2\hbar}\left(x_a^2 + x_b^2\right)$$

$$A(\tau) = \left[\frac{m\omega}{2\pi\hbar\sinh(\omega\tau)}\right]^{1/2} \xrightarrow[\tau\to\infty]{} \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\omega\tau/2}$$

comparing both sides in limit

$$\langle x_b|0\rangle\langle 0|x_a\rangle e^{-\tau E_0/\hbar} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega x_b^2}{2\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x_a^2}\, e^{-\omega\tau/2}$$

giving:

$$E_0 = \frac{\hbar\omega}{2} \qquad \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega x^2}{2\hbar}}$$