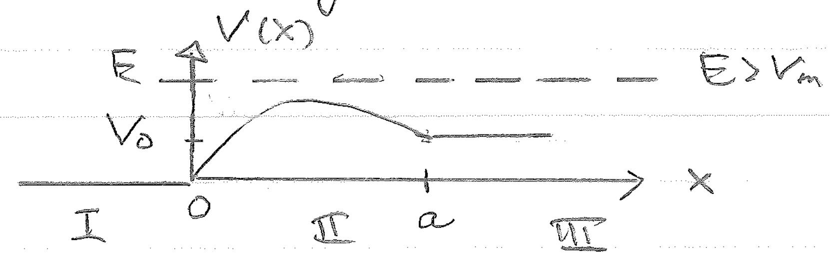
Physics 491

Lecture #6 : 1D Scattering

For $V(x)$ time independent, plane wave, time independent scattering



$V = 0 \quad x < 0$ free particle $k = \sqrt{2mE}/\hbar$

$V(x) \quad 0 < x < a$ barrier

$V = V_0 \quad x > a$ free particle $k' = \sqrt{2m(E - V_0)}/\hbar$

Assuming particle incident from left

$\psi_I = Ae^{ikx} + Be^{-ikx}$ $\qquad e^{+i(kx - \omega t)}$ right moving

$\psi_{III} = Ce^{ik'x}$ $\qquad e^{-ikx - \omega t}$ left moving

plane wave probability current

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) \propto \frac{\hbar k}{m}$$

incident $j_i = |A|^2 \frac{\hbar k}{m}$

reflected $j_r = |B|^2 \hbar k/m$ defined as absolute value of current

transmitted $j_t = |C|^2 \frac{\hbar k'}{m}$

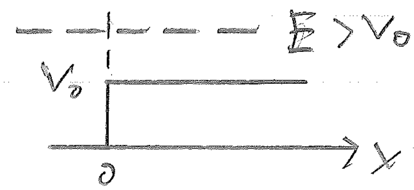
Reflected, transmitted probabilities

$$R = j_r / j_i = \left|\frac{B}{A}\right|^2$$

$$T = j_t / j_i = \left|\frac{C}{A}\right|^2 \frac{k'}{k}$$

probability conservation $R+T=1$
Without loss of generality can take $A=1$

Step-up potential



$$\psi_1 = e^{ikx} + Be^{-ikx}$$
$$\psi_2 = C e^{+ik'x}$$

(i) $\psi_1(0) = \psi_2(0)$ $\qquad 1 + B = C$

(ii) $\psi_1'(0) = \psi_2'(0)$ $\qquad ik(1-B) = ik'C$

$$C = \frac{2k}{k+k'} \qquad B = \frac{k-k'}{k+k'}$$

$$T = \frac{k'}{k}|C|^2 = \frac{4kk'}{(k+k')^2} \qquad R = \left(\frac{k-k'}{k+k'}\right)^2$$

easily check $T+R=1$

Barrier penetration $E < V_0$

$k' = iq$ q real

$q = \sqrt{2m(V_0 - E)}/\hbar$

$\psi_2 = C e^{-qx}$ exponential decay in classically forbidden region

What about T? $C \stackrel{?}{=} \frac{2k}{k+iq} \neq 0$

Go back to definition of j. For ψ real, $j = 0$ so $T = 0$

penetration depth $|\psi_2|^2 = |C|^2 e^{-2qx}$

$$\lambda = \frac{1}{2q} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}$$

<u>The S-matrix</u> the key idea in general theory of scattering introduced here in simple context.

Consider δ-function potential (b, a length)

$$V(x) = -\frac{\hbar^2}{2m}\left(\frac{1}{b}\right)\delta(x) \quad , \quad k^2 = 2mE/\hbar^2$$

$$\dim[k] = \frac{1}{\text{length}}$$

$$\psi'' + \frac{1}{b}\delta(x)\psi = -k^2\psi$$

$$\dim\left[\frac{\hbar^2}{2m}\right] = \text{energy} \times \text{length}^2$$

General solution (particle incident from left or right)

$$\psi_- = Ae^{ikx} + Be^{-ikx} \qquad x<0$$
$$\psi_+ = Fe^{ikx} + Ge^{-ikx} \qquad x>0$$

Boundary condition

$$\psi_-(0) = \psi_+(0)$$

$$\left.\frac{d\psi_+}{dx}\right|_0 - \left.\frac{d\psi_-}{dx}\right|_0 = -\frac{1}{b}\psi_+(0)$$

$$A+B = F+G$$
$$ik(F-G) - ik(A-B) = -\frac{1}{b}(A+B)$$

multiply 2^{nd} equation by $\frac{i}{k}$

$$A-B-(F-G) = \frac{-i}{kb}(A+B)$$

$$A\underbrace{\left(1+\frac{i}{kb}\right)}_{\equiv \alpha} - B\underbrace{\left(1-\frac{i}{kb}\right)}_{=\alpha^*} = F - G$$

then

$$A + B = F + G$$
$$\alpha A - \alpha^* B = F - G$$

A, G are left/right incident wave amplitudes
B, F are scattered wave amplitude

$$B - F = -A + G$$
$$-\alpha^* B - F = -\alpha A - G$$

$$\begin{pmatrix} 1 & -1 \\ \alpha^* & 1 \end{pmatrix}\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} A \\ G \end{pmatrix}$$

recall inverse of 2×2 matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$[A^{-1}] = \frac{1}{a_{11}a_{22} - a_{21}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Define S matrix as

$$\begin{pmatrix} B \\ F \end{pmatrix} = [S]\begin{pmatrix} A \\ G \end{pmatrix}$$

In the δ-function case

$$[S] = \begin{pmatrix} 1 & -1 \\ \alpha^* & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ \alpha & 1 \end{pmatrix}$$

the S-matrix will <u>diverge</u> when $1+\alpha^* = 0$

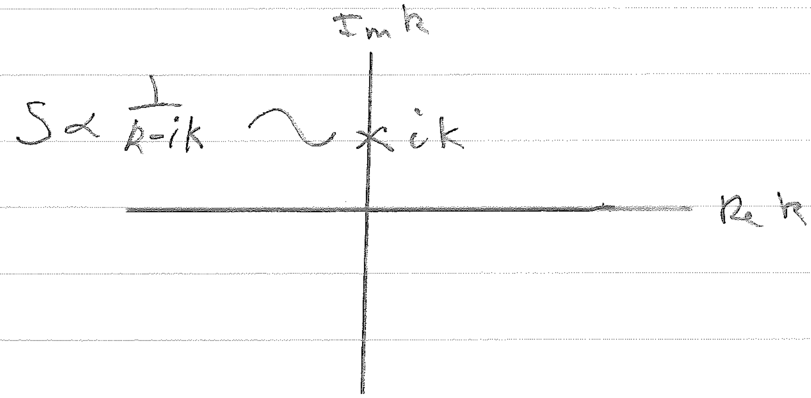
$$1+\alpha^* = 1 + 1 - \frac{i}{kb} = 0$$

$$k = \frac{i}{2b} \equiv iK$$

This value of k corresponds to <u>bound state</u>:

$$E = \frac{\hbar^2}{2m}\left(\frac{i}{2b}\right)^2 = \frac{-\hbar^2}{8b} \qquad \text{(lecture \#5)}$$

Generally, S-matrix is analytic function of k except for simple pole corresponding to bound state, $S \propto \frac{1}{k-iK}$



More careful treatment shows S-matrix <u>resonance</u> corresponds to bound state with <u>finite</u> lifetime

Also, $\hat{S}^{+}\hat{S} = \hat{1}$ (unit operator or matrix)

ensures probability conservation.

Returning to S-matrix for δ-function

$$[S] = \begin{pmatrix} 1 & -1 \\ \alpha^* & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ \alpha & 1 \end{pmatrix} = \frac{1}{1+\alpha^*} \begin{pmatrix} 1 & 1 \\ -\alpha^* & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ \alpha & 1 \end{pmatrix}$$

$$= \frac{1}{1+\alpha^*} \begin{pmatrix} -1+\alpha & 2 \\ \alpha^*+\alpha & 1-\alpha^* \end{pmatrix}$$

with $\alpha = 1 + \frac{i}{kb}$ and $kb = X$

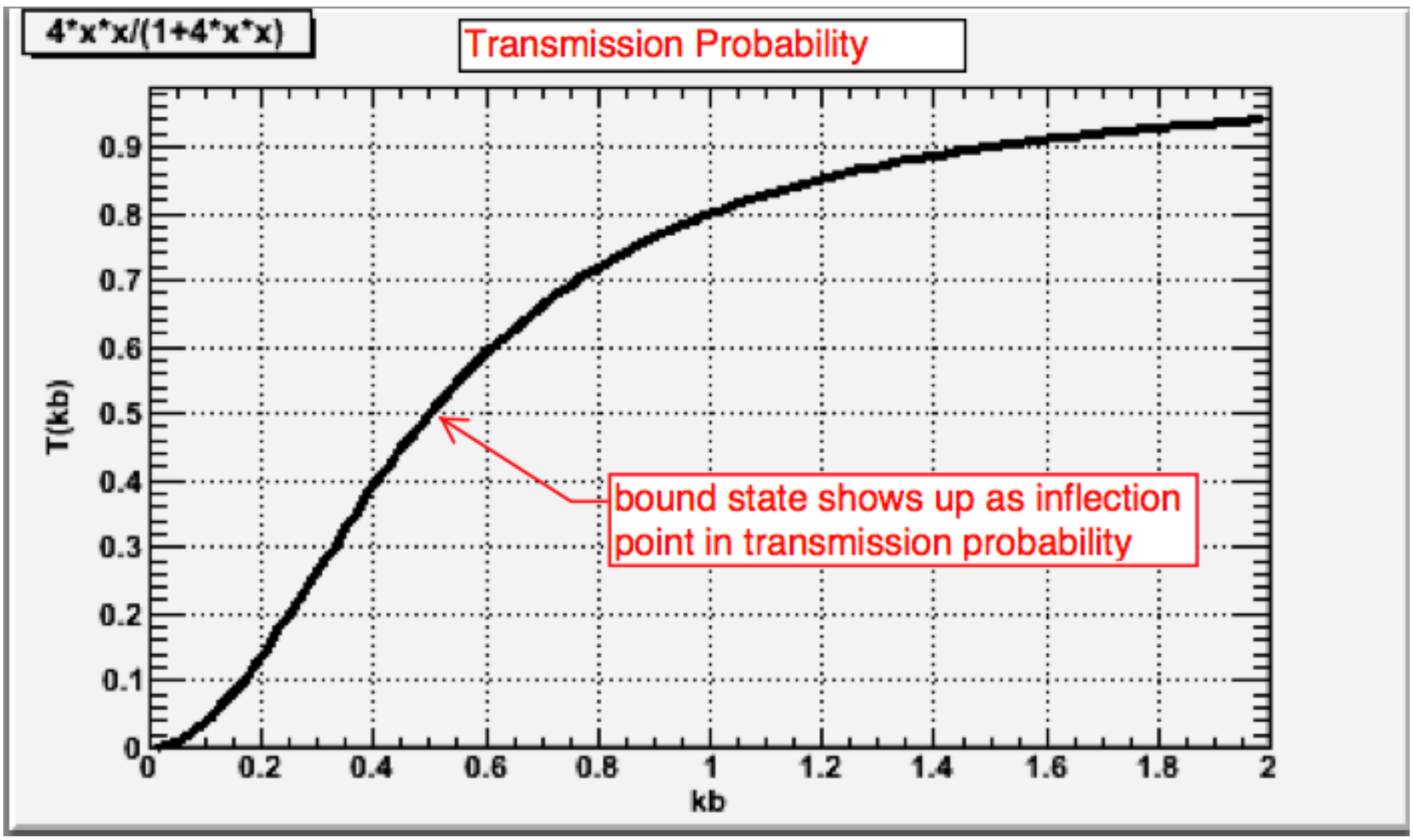
$$[S] = \frac{1}{2X-i} \begin{pmatrix} i & 2X \\ 2X & i \end{pmatrix}$$

for particle incident from left

$$\begin{pmatrix} B \\ F \end{pmatrix} = [S] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2X+i} \begin{pmatrix} i \\ 2X \end{pmatrix}$$

$$R = |B|^2 = \frac{1}{1+4X^2}$$

$$T = |F|^2 = R-1 = \frac{4X^2}{1+4X^2}$$


4*x*x/(1+4*x*x)
Transmission Probability
T(kb)
kb
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
bound state shows up as inflection
point in transmission probability