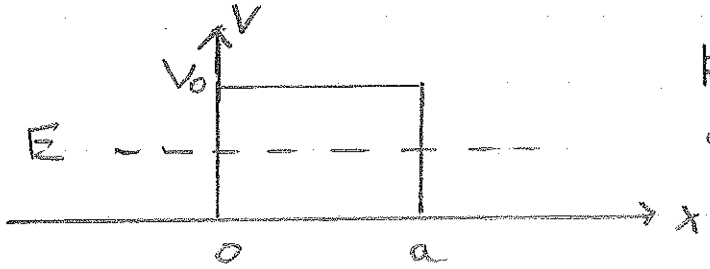
 physics 491

Lecture #7: Tunneling



$k = \sqrt{2mE}/\hbar$

$q = \sqrt{2m(V_0 - E)}/\hbar$

Wave incident from left. Include reflections at $x=0$ and a.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Fe^{qx} + Ge^{-qx} & 0<x<a \\ Ce^{ikx} & x>a \end{cases}$$

$$T = \frac{j_t}{j_{in}} = \frac{\frac{\hbar k}{m}|C|^2}{\frac{\hbar k}{m}|A|^2} = \left[1 + \left(\frac{k^2+q^2}{2kq}\right)^2 \sinh^2 qa\right]^{-1}$$

for $qa \gg 1$, $\sinh qa = \frac{e^{qa} - e^{-qa}}{2} \simeq \frac{e^{qa}}{2}$

$$T \xrightarrow[qa\gg 1]{} \left(\frac{4kq}{k^2+q^2}\right)^2 e^{-2qa}$$

<u>macroscopic</u> $V_0 - E = 1\,\text{erg}$ $a = 1\,\text{cm}$ $m = 1\,\text{g}$

$qa \sim 10^{27}$!

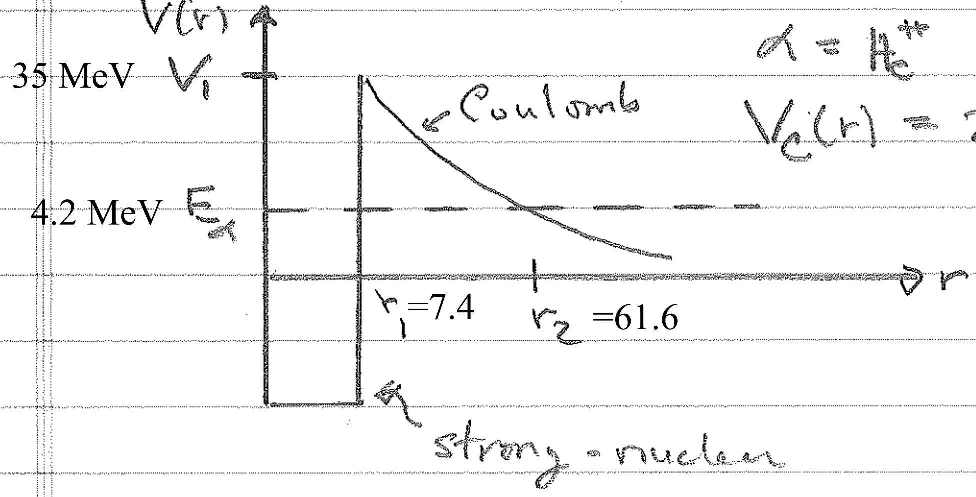
<u>e^- in metal</u> $V_0 - E \sim 10\,\text{eV}$ $a = 1\,\text{nm}$

$$qa = \frac{\sqrt{2mc^2(V_0-E)}}{\hbar c}a = \frac{\sqrt{10^6\,\text{eV}\,(10\,\text{eV})}\,(1\,\text{nm})}{200\,\text{eV}\cdot\text{nm}} \sim 10$$

Example: α decay of heavy nucleus

Gamov's theory, see Griffiths 8.2

values for A=238



$\alpha = He^{++}$ $\quad Z_d = Z - 2$

$V_c(r) = 2 Z_d \, e^2/r$

$4 Z_\alpha$

$E = 2 Z_d \, e^2/r_2$; $V_1 = 2 Z_d \, e^2/r_1$

$r_1 \cong (1.2 \text{ fm}) \, A^{1/3}$ = 7.4 fm, A=238

for U_{238}, $Z = 92$, $Z_d = 90$. U-238 decays to Th-234 + alpha

$E_\alpha = 4.2$ MeV experimental value

calculate V_1 from r_1

$$V_1 = 2(90)\left(\frac{e^2}{\hbar c}\right)\frac{\hbar c}{r_1} = 2(90)\frac{1}{137}\frac{197 \text{ MeV}\cdot\text{fm}}{7.4 \text{ fm}}$$

$$= 35.0 \text{ MeV}$$

calculate r_2 from experimental alpha value

$$V_2 = V(r_2) = E_\alpha = 2 Z_d \frac{\hbar c \alpha}{r_2}$$

$$r_2 = 2 Z_d \alpha \frac{\hbar c}{E_\alpha} = 2(90)\frac{1}{137}\frac{197 \text{ MeV}\cdot\text{fm}}{4.2 \text{ MeV}}$$

$$= 61.6 \text{ fm}$$

First approximation, Square barrier

$$T = \frac{4E(V_0 - E)}{V_0^2} e^{-2\gamma a}$$

E=4.5 MeV

take $V_0 \cong \frac{1}{2} V_1 = 17.5$ MeV

$a \cong \frac{1}{2}(r_2 - r_1) = 26.5$ fm

$$\frac{4E(V_0 - E)}{V_0^2} = \frac{4(\overset{4.5}{\text{MeV}})(13.5\ \text{MeV})}{(17.5\ \text{MeV})^2} = 0.79 \cong 1$$

typically, this energy pre-factor is ≈ 1.

$$\gamma = \frac{\sqrt{2m_\alpha c^2 (V_0 - E)}}{\hbar c} = \frac{[2(4)\overset{\text{amu}}{931.5\ \text{MeV}}(13.5\ \text{MeV})]^{1/2}}{197\ \text{MeV}\cdot\text{fm}}$$

$$= (0.62\ \text{fm})^{-1}$$

$$\gamma a = \frac{26.5\ \text{fm}}{0.62\ \text{fm}} = 42.7$$

$$T = e^{-2(42.7)} = 10^{-85.4 \log_{10} e} = 10^{-37}$$

Experimental quantity is decay rate $= \tau^{-1}$!
We need a characteristic frequency.
Gamov make a classical estimate based on the frequency with which the α-particle strikes the barrier.

$$f = \frac{v}{2r_1} = \frac{c}{2r_1}\sqrt{\frac{2E_\alpha}{m_\alpha c^2}} = \frac{c}{2r_1}\sqrt{\frac{2(4.2\ \text{MeV})}{4(931.5\ \text{MeV})}}$$

$$= 0.0475\ c/2(7.4\ \text{fm}) = 3\times10^{23}\ \text{fm/s}(0.0475)/2/7.4\ \text{fm}$$

7.4

$$\boxed{f = 9.6 \times 10^{20} s^{-1}}$$

1/e lifetime $\tau = (f T)^{-1} = 10^{16} s$

$$= \frac{10^{16} s}{\pi \times 10^{7} s/y} \approx 0.3 \, Gy$$

exp value $= 6.45$ Gy

$$\tau_{\frac{1}{2}}^{exp} = \ln 2 \, \tau = 4.47 \, Gy$$

Remarks:

(i) exponential dependence account for enormous range of lifetime

(ii) need better approximation

$$g a \rightarrow \int dr \, g(r) \equiv I$$

integration over classically forbidden region

$$T = \exp(-2I)$$

justified by WKB approximation Griffiths ch 8.

Semi-classical Griffiths P8.2 (p332)

expand in power of $\hbar$

$$\psi(x) = e^{if/\hbar}$$

$$p(x) \equiv 2m(E - V(x))$$

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi$$

$$\hbar^2 \psi'' = -p^2 \psi \qquad (1)$$

$$\psi' = \frac{i}{\hbar} f' e^{if/\hbar}$$

$$\psi'' = \frac{i}{\hbar} f'' e^{if/\hbar} + \left(\frac{i}{\hbar}\right)^2 (f')^2 e^{if/\hbar}$$

(1) becomes

$$i\hbar f'' - (f')^2 + p^2 = 0$$

Expand as

$$f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) + \cdots$$

Keep $O(\hbar)^1$ terms

$$i\hbar(f_0'' + \hbar f_1'') - (f_0' + \hbar f_1')^2 + p^2 = 0$$

to first order in $\hbar$

$(\hbar)^0$: $\quad -(f_0')^2 + p^2 = 0$

$$f_0' = \pm p$$

$$f_0(x) = \pm \int^x p(x')\,dx'$$

$(\hbar)^1$: $\quad i f_0'' - 2 f_0' f_1' = 0$

$$f_1' = \frac{i}{2}\,\frac{f_0''}{f_0'} = \frac{i}{2}\left(\frac{\pm p'}{\pm p}\right)$$

$$= \frac{i}{2}\frac{d}{dx}\ln(\pm p)$$

$$f_1 = \frac{i}{2}\ln(\pm p/c)$$

$$= \frac{i}{2}\ln\left(\frac{p}{c}\right) \quad (+)$$

$$\frac{i}{2}\ln\left(\frac{p}{c}\right) + i n\pi \quad (-)$$

$$\psi = e^{\frac{i}{\hbar}(f_0 + \hbar f_1)} = e^{\frac{i}{\hbar} f_0}\, e^{i f_1}$$

just a phase so ignore

$$\psi = e^{\frac{i}{\hbar} f_0}\, e^{\ln\left(\frac{p}{c}\right)^{-1/2}} = \sqrt{\frac{c}{p}}\, e^{\frac{i}{\hbar}\pm\int p(x)\,dx}$$

$$\psi = \frac{C'}{\sqrt{p}}\, e^{\frac{i}{\hbar}\pm\int^x p(x')\,dx'}$$

Z_d 5

Integral over classically forbidden region:

$$q(r) = \frac{\sqrt{2mE}}{\hbar}\sqrt{\frac{V(r)}{E} - 1} = \frac{\sqrt{2mE}}{\hbar}\sqrt{\frac{r_2}{r} - 1}$$

$$I = \int_{r_1}^{r_2} dr\, q(r) = \frac{\sqrt{2mE}}{\hbar}\left[r_2 \cos^{-1}\sqrt{\frac{r_1}{r_2}} - \sqrt{r_1(r_2 - r_1)}\right]$$

for $r_1 \ll r_2$; $\cos^{-1} x \cong \frac{\pi}{2} - x$; $r_1(r_2 - r_1) \cong r_1 r_2$

x=√(r_1/r_2)

$$I = \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2} r_2 - 2\sqrt{r_1 r_2}\right]$$

with $r_2 = \frac{1}{E} 2(Z_d)\hbar c \alpha$ $\quad \propto \sqrt{Z_d}^{4/3}$

because r_1 goes like Zd^(1/3)

$$I = K_1\left[\frac{Z_d}{\sqrt{E}} - \frac{K_2}{K_1}\sqrt{r_1 Z_d}\right]$$

$$K_1 = \alpha\pi\sqrt{2m_\alpha c^2} = 1.980\,(\text{MeV})^{1/2}$$

$$K_2 = 4\sqrt{\frac{\alpha m c^2}{\hbar c}} = 1.486\,(\text{fm})^{-1/2}$$

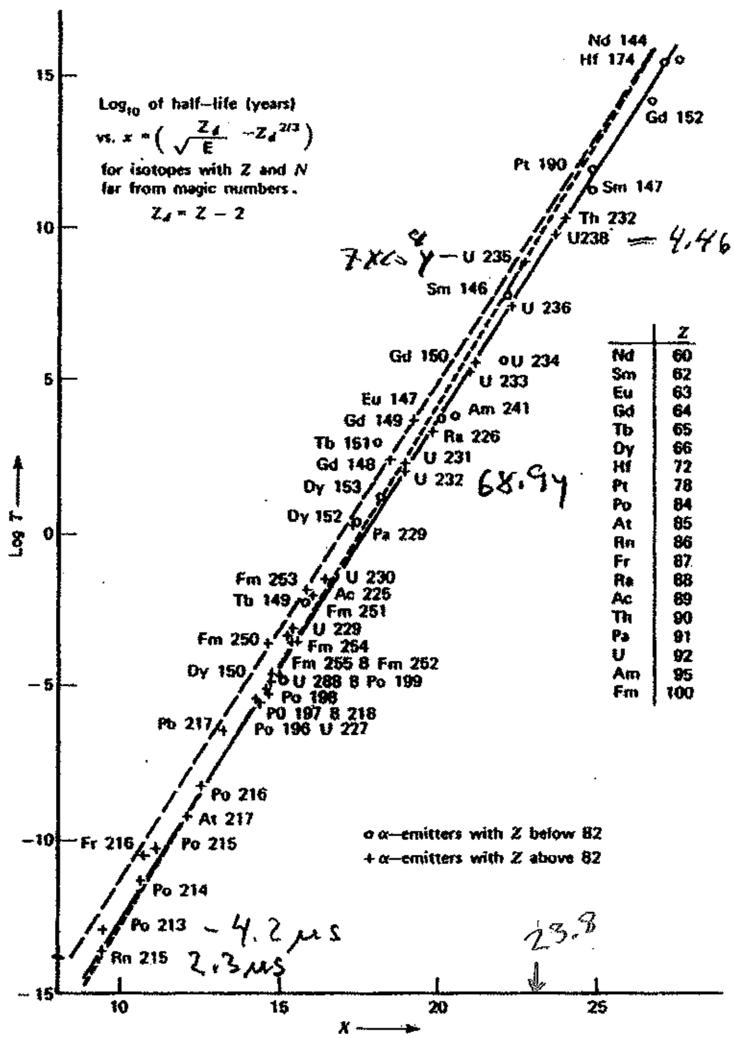
for large A, $A \approx 2Z_d$

$\frac{K_2}{K_1} = 0.751\,(\text{MeV}\cdot\text{fm})^{-1/2}$; $r_1 = 1.2\,\text{fm}\, A^{1/3} \cong \boxed{1.5\,\text{fm}}\, Z_d^{1/3}$

$$\boxed{I = K_1\left[\frac{Z_d}{\sqrt{E/\text{MeV}}} - Z_d^{2/3}\right]} \cong K_1 X$$

where the factor $\frac{K_2}{K_1}\left(\boxed{1.5\,\text{fm}}\right)^{1/2} = 0.92\,\text{MeV}^{-1/2} \approx 1\,\text{MeV}^{-1/2}$

α-decay data: Log_{10} of $\frac{1}{2}$ life (years) vs
$X = \frac{Z_d}{\sqrt{E}} - Z_d^{2/3}$ where $Z_d = Z-2$ (daughter)



	Z
Nd	60
Sm	62
Eu	63
Gd	64
Tb	65
Dy	66
Hf	72
Pt	78
Po	84
At	85
Rn	86
Fr	87
Ra	88
Ac	89
Th	90
Pa	91
U	92
Am	95
Fm	100

$X_{U238} = 23.8\ \mathrm{MeV}^{-1/2}$

From: Hyde, Perlman and Seaborg
The Nuclear Properties of Heavy Elements, Vol. I
Prentice-Hall, Englewood Cliffs, NJ (1964)

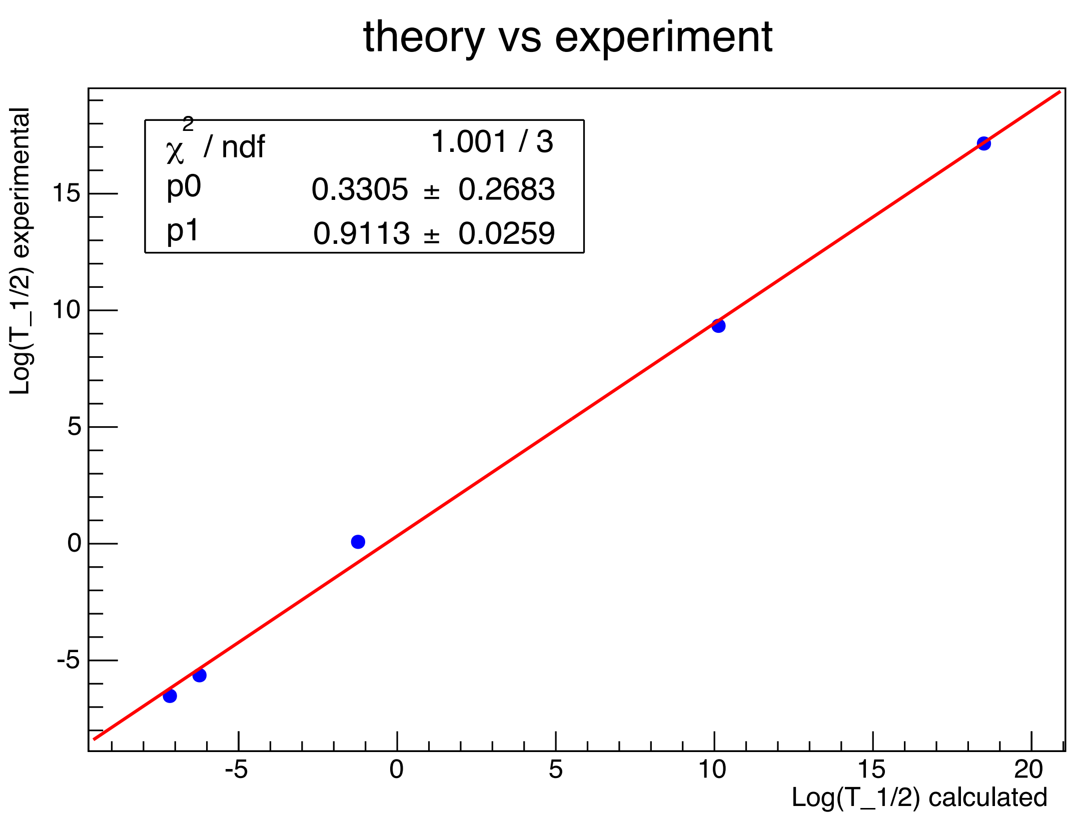
From PDG:

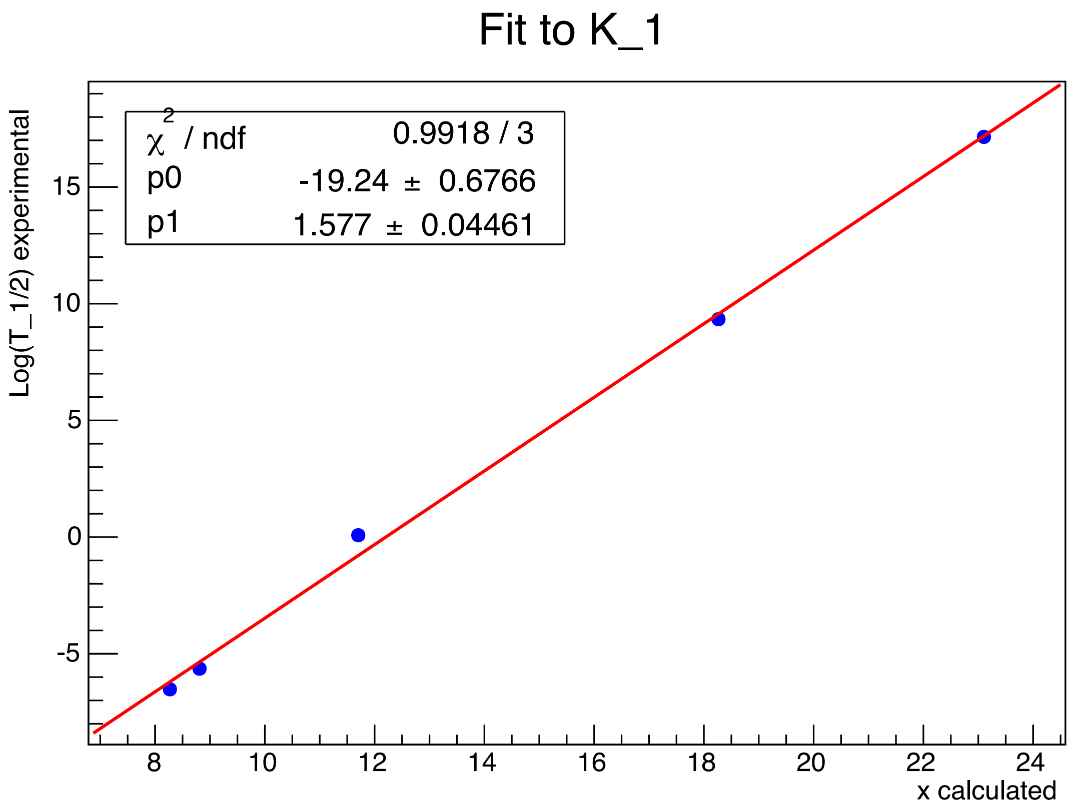
	$T_{1/2}$
U^{238}	4.47 Gy
Po^{212}	0.299 μs $= 3\times10^{-7}$ s

Date: September 3, 2015 at 3:08 PM
Topic: alpha decay calculation

<u>nucleus A Z Q half-life computed (experimental)</u>
U-238,A=238, Z=92, Q=4.270 MeV T_1.2 = 3.148E+18 (1.410E+17) seconds
U-232,A=232, Z=92, Q=5.413 MeV T_1.2 = 1.349E+10 (2.173E+09) seconds
Th-215,A=215, Z=86, Q=8.839 MeV T_1.2 = 5.831E-07 (2.300E-06) seconds
Po-212,A=212, Z=84, Q=8.954 MeV T_1.2 = 6.796E-08 (3.000E-07) seconds
Rn-215,A=215, Z=90, Q=7.666 MeV T_1.2 = 5.834E-02 (1.200E+00) seconds

Comparison between this calculation and experimental values:




Fit to K_1
χ^2 / ndf 0.9918 / 3
p0 -19.24 ± 0.6766
p1 1.577 ± 0.04461
Log(T_1/2) experimental
x calculated
15
10
5
0
-5
8
10
12
14
16
18
20
22
24