Physics 491
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Lecture #9: Spin-$\frac{1}{2}$

Observables are eigenvalues of Hermitian operators.

Theorem: Eigenvalues of Hermitian operators are real, and converse

label state by eigenvalue

$$\hat{A}|a_i\rangle = a_i|a_i\rangle \quad ; \quad \langle a_i|\hat{A}^\dagger = \langle a_i|a_i^*$$

$$\langle a_i|(\hat{A}^\dagger - \hat{A})|a_j\rangle = (a_i^* - a_j)\langle a_i|a_j\rangle$$
$$= (a_i^* - a_i)\,\delta_{ij}$$

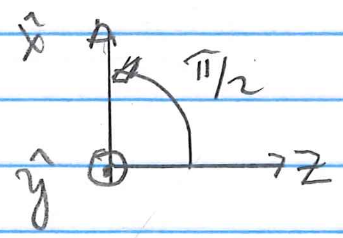
can replace j by i because of δ_{ij}

ie $\hat{A}^\dagger = \hat{A}$ iff $a_i^* = a_i$

Change of basis We introduced 3 linearly independant bases for spinor state $|\chi\rangle$:

$$|+z\rangle, |-z\rangle \; ; \; |+x\rangle, |-x\rangle \; ; \; |+y\rangle, |-y\rangle$$

bases are related by a rotation in physical space.



right handed $\pi/2$ rotation about $\hat{y}$

Corresponding operator on spinor states,

spinor

$$\hat{R}\left(\tfrac{\pi}{2}\hat{y}\right)|\pm z\rangle = |\pm x\rangle$$

Symmetry is invariance under transformation

In Q.M., invariant is state norm. For spinors,

$$\langle \chi|\chi\rangle = (\chi_1^*, \chi_2^*)\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = |\chi_1|^2 + |\chi_2|^2 = 1$$

transformations are unitary operators

$$\hat{U}^+\hat{U} = \hat{I} \qquad |\chi'\rangle = \hat{U}|\chi\rangle$$
$$\langle\chi'| = \langle\chi|\hat{U}^+$$

so $\langle\chi'|\chi'\rangle = \langle\chi|\hat{U}^+\hat{U}|\chi\rangle = \langle\chi|\chi\rangle$

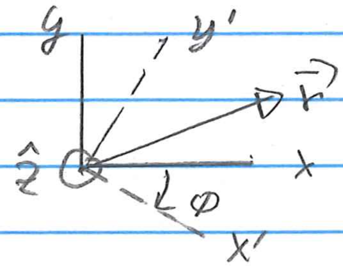
Analogous to invariance of $\vec{r}\cdot\vec{r}$ under rotation
and Lorentz 4-vector norm
$(ct)^2 - \vec{r}\cdot\vec{r}$ under Lorentz transformation
Worth taking a moment to define group.

Group is a set $G = \{g_i\}$ of elements
with rule called group product $g_1 \cdot g_2 = g_3$

(i) $f, g \in G$ then $h = f \cdot g \in G$
(ii) $(f \cdot g)\cdot h = f\cdot(g \cdot h)$
(iii) existence of identity I
(iv) existence of inverse $g^{-1} \cdot g = I$
not commutative in general, $f \cdot g \neq g \cdot f$

Change of basis (passive rotation)

for example Euclidean vector in plane.



rotation by $-\phi$ about $\hat{z}$

$$(\hat{x}', \hat{y}') = (\hat{x}, \hat{y}) \underbrace{\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}}_{R^E(-\phi\hat{z})}$$

$$\vec{r} = x\hat{x} + y\hat{y} = x'\hat{x}' + y'\hat{y}'$$

$$= (\hat{x}, \hat{y}) \underbrace{R^E(-\phi\hat{z}) R^E(\phi\hat{z})}_{\text{Identity}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (\hat{x}', \hat{y}') \begin{pmatrix} x' \\ y' \end{pmatrix}$$

<u>Components</u> transform <u>inversely</u> to <u>basis</u>

Spinor change of basis operator $\hat{S}$ is unitary.
general spinor basis is 2 dimentional

$$|a_1\rangle, |a_2\rangle \quad \text{with } \langle a_i|a_j\rangle = \delta_{ij}$$

consider change to another basis as

$$|b_i\rangle = \hat{S}|a_i\rangle$$

$$|b_i\rangle = \sum_{j=1}^{2} |a_j\rangle \underbrace{\langle a_j|\hat{S}|a_i\rangle}_{[\hat{S}]^a_{ji}}$$

transformation of basis

$[\hat{S}]^a_{ji}$ matrix in a-basis

Notice summation over <u>row</u> index implies matrix multiplication on right

$$\Big(|b_1\rangle, |b_2\rangle\Big) = \Big(|a_1\rangle, |a_2\rangle\Big)\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^a$$

<u>Spinor</u> <u>components</u> transform as

$$|\chi\rangle = \sum_i |a_i\rangle\langle a_i|\chi\rangle$$

$$= \sum_{i,j} |b_j\rangle\langle b_j|a_i\rangle\langle a_i|\chi\rangle$$

with $\langle b_j| = \langle a_j|\hat{S}^+$ we get

$$|\chi\rangle = \sum_j |b_j\rangle\Big(\sum_i \langle a_j|\hat{S}^+|a_i\rangle\langle a_i|\chi\rangle$$

then

$$\langle b_k|\chi\rangle = \sum_j \underbrace{\langle b_k|b_j\rangle}_{\delta_{kj}} \sum_i \langle a_j|\hat{S}^\dagger|a_i\rangle\langle a_i|\chi\rangle$$

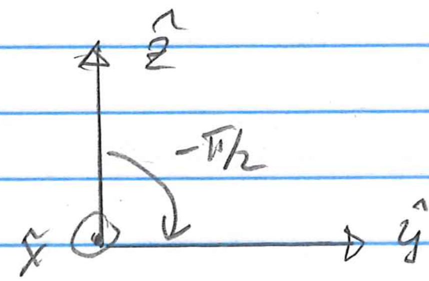
$$= \sum_i \langle a_k|\hat{S}^\dagger|a_i\underbrace{\rangle\langle}_{\text{summed over column index}} a_i|\chi\rangle$$

or in component form:

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}^b = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^{a\,\dagger} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}^a$$

$$= \begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix}^a \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}^a$$

Example



$$|\pm y\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle \pm i|-z\rangle\right)$$

$$\left(|+y\rangle, |-y\rangle\right) = \left(|+z\rangle, |-z\rangle\right) \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{R^S(-\frac{\pi}{2}\hat{x})}$$

Components transform as $R^S(-\frac{\pi}{2}\hat{x})^\dagger = R^S(+\frac{\pi}{2}\hat{x})$

$$\xrightarrow[z]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}^z$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^y = R^S(+\tfrac{\pi}{2}\hat{x}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^z$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

giving

$$|+y\rangle \xrightarrow[y]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \; |-y\rangle \xrightarrow[y]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>transformation of operators</u>

let $|b_i\rangle = \hat{S}|a_i\rangle$

$$\langle b_i|\hat{O}|b_j\rangle = \langle a_i|\hat{S}^\dagger \hat{O} \hat{S}|a_j\rangle$$

$$= \sum_{k,l} \langle a_i|\hat{S}^\dagger|a_k\rangle\langle a_k|\hat{O}|a_l\rangle\langle a_l|\hat{S}|a_j\rangle$$

in matrix form

$$[\hat{O}]^b_{ij} = [\hat{S}^\dagger]^a_{ik}\,[\hat{O}]^a_{kl}\,[\hat{S}]^a_{lj}$$

or $\quad \hat{O}' = \hat{S}^\dagger \hat{O} \hat{S}$

<u>example</u> change from $z \to y$ basis

$$\hat{S} = R^S\left(-\frac{\pi}{2}\hat{x}\right) \qquad \hat{S}^+ = R^S\left(+\frac{\pi}{2}\hat{x}\right) \underset{z}{\Rightarrow} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}$$

$$\left[\hat{S}_y\right]^y = \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \frac{\hbar}{2}\,\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^y$$

just as we expect.