Physics 491
Fall 2015

Recitation #11 Solutions

#1) Writing $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \frac{1}{2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$ where $\hat{S}^2 = \left(\hat{\vec{S}}_1 + \hat{\vec{S}}_2\right)^2$ we find

$$\hat{S}_1 \cdot \hat{S}_2 |1,m\rangle = \frac{\hbar^2}{2}\left(1(1+1) - 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)\right)|1,m\rangle$$

$$= \frac{\hbar^2}{4}|1,m\rangle$$

$$\hat{S}_1 \cdot \hat{S}_2 |0,0\rangle = \frac{\hbar^2}{2}\left(0 - 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)\right) = -\frac{3\hbar^2}{4}|0,0\rangle$$

$|1,m\rangle$ states are degenerate due to rotational invariance of $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ operator.

#2) $|3/2, 3/2\rangle = |{+}{+}{+}\rangle$

$$\hat{S}_- |3/2, 3/2\rangle = \hbar\sqrt{3/2(3/2+1) - 3/2(3/2-1)}\,|3/2, 1/2\rangle = \sqrt{3}\hbar\,|3/2, 1/2\rangle$$

$$= \left(\hat{S}_-^1 + \hat{S}_-^2 + \hat{S}_-^3\right)|{+}{+}{+}\rangle = \hbar\left[|{-}{+}{+}\rangle + |{+}{-}{-}\rangle + |{+}{+}{-}\rangle\right]$$

$$\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right) = 1$$

$$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}}\left[|{-}{+}{+}\rangle + |{+}{-}{+}\rangle + |{+}{+}{-}\rangle\right]$$

$$\hat{S}_- |3/2, 1/2\rangle = \hbar\sqrt{3/2(3/2+1) - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)} = 2\hbar\,|3/2, -1/2\rangle$$

applying $\hat{S}_-^1 + \hat{S}_-^2 + \hat{S}_-^3$ we get two of each double-minus state canceling this 2 to get

Recitation #11

$$|3/2, -1/2\rangle = \frac{1}{\sqrt{3}}\left[|--+\rangle + |-+-\rangle + |+--\rangle\right]$$

$$\hat{S}_-|3/2, -\tfrac{1}{2}\rangle = \hbar\sqrt{3/2(3/2+1) + 1/2(-\tfrac{1}{2}-1)}\,|3/2, -3/2\rangle$$

$$= \sqrt{3}\hbar\,|3/2, -3/2\rangle$$

$$(\hat{S}_-^1 + \hat{S}_-^2 + \hat{S}_-^3)|3/2, 1/2\rangle = \frac{1}{\sqrt{3}}\left(3|---\rangle\right)$$

$$|3/2 - 3/2\rangle = |---\rangle.$$

#3. Using the b-basis, $2\frac{(\vec{b}\cdot\vec{S})}{\hbar}|\tfrac{1}{2}, m\rangle = 2bm|\tfrac{1}{2}, m\rangle$

So eigenvalues of $\hat{A}$ are $a \pm b$ for $m = \pm\frac{1}{2}$.

Using the taylor series expansion,

$$f(\hat{A})|\tfrac{1}{2}, m\rangle = f(a+2bm)|\tfrac{1}{2}, m\rangle$$

eigenvalues are $f(a\pm b)$. So

$$f(\hat{A})|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = f(a\pm b)|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = (\alpha \pm \beta)|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$

$$\alpha = \tfrac{1}{2}\left[f(a+b) + f(a-b)\right]$$

$$\beta = \tfrac{1}{2}\left[f(a+b) - f(a-b)\right]$$