Physics 491

Recitation # 4 Solution

not assigned

"fun with the δ-function"

① prove $\langle k' | k \rangle = \delta(k'-k)$

$$\langle k' | k \rangle = \int \langle k' | x \rangle \langle x | k \rangle \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} \, dx = \delta(k'-k)$$

② prove $\delta(x) = \delta(-x)$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk \qquad \text{let } k \to -k'$$

$$dk \to -dk'$$

but $-\infty \Leftrightarrow +\infty$ interchange gives another minus sign so

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, dk = \delta(-x)$$

③ Prove $\frac{\partial}{\partial x}\delta(x'-x) = -\frac{\partial}{\partial x'}\delta(x-x')$

$$\frac{\partial}{\partial x}\delta(x'-x) = \frac{\partial}{\partial x}\left(\frac{1}{2\pi}\right)\int e^{ik(x-x')}\,dk = \frac{1}{2\pi}\int ik\,e^{ik(x-x')}\,dk$$

$$\frac{\partial}{\partial x'}\delta(x'-x) = \frac{\partial}{\partial x'}\left(\frac{1}{2\pi}\right)\int e^{ik(x-x')}\,dk = \frac{1}{2\pi}\int(-ik)\,e^{ik(x-x')}\,dk$$

$$= -\frac{\partial}{\partial x}\delta(x'-x)$$

④ Prove $\frac{\partial}{\partial x}\delta(x'-x) = \delta(x'-x)\frac{\partial}{\partial x'}$

$$f(x) = \int \delta(x'-x)\,f(x')\,dx'$$

$$\frac{\partial f}{\partial x} = \int \frac{\partial}{\partial x}\delta(x'-x)\;f(x')\,dx'$$

$$= -\int \frac{\partial}{\partial x'}\delta(x'-x)\,f(x')\,dx'$$

from ③

$$= +\int \delta(x'-x)\,\frac{\partial}{\partial x'}f(x')\,dx'$$

integrate by parts