

Recitation #6 Solutions

① for the spin state

$$|\psi\rangle = A \left[ 3i |+\rangle + 4 |-\rangle \right] \xrightarrow{+z} A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$\langle\psi|\psi\rangle = 1 = A^2(3^2 + 4^2) \Rightarrow A = \frac{1}{5}$$

$$\begin{aligned} \langle \hat{S}_y \rangle &= \frac{1}{5^2} (-3i, 4) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= \frac{\hbar}{2} \frac{1}{5^2} (-3i, 4) \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{2} \frac{1}{5^2} 12(-1-1) \\ &= -\hbar^2 \frac{12}{25} \end{aligned}$$

$$\hat{S}_y = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \langle S_y^2 \rangle = \frac{\hbar^2}{4}$$

$$\begin{aligned} (\Delta S_y)^2 &= \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \hbar^2 \left( \frac{12}{25} \right)^2 = +0.02\hbar^2 \\ &\quad \underbrace{(0.48)^2}_{= 0.23} \end{aligned}$$

$$\Delta S_y = +0.14\hbar$$

$$\begin{aligned} \textcircled{2} \quad \hat{P}_+ &= |+\rangle\langle +| \xrightarrow{+y} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \hat{P}_- &= |-\rangle\langle -| \xrightarrow{+y} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$(|+\rangle, |-\rangle) = (|+\rangle, |-\rangle) \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{\hat{S}}$$

$$\begin{aligned} [\hat{P}_+]^y &= \hat{S}^\dagger [\hat{P}_+]^z \hat{S} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \end{aligned}$$

$$\begin{aligned} [\hat{P}_-]^y &= \hat{S}^\dagger [\hat{P}_-]^z \hat{S} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ i & -i \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -i & i \end{pmatrix} \end{aligned}$$

$$|+\rangle \xrightarrow{+y} \hat{S}^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\rangle \xrightarrow{+y} \hat{S}^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-i}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{P}_+ |+\rangle \xrightarrow{+y} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \left(\frac{1}{2}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{P}_- |+\rangle \xrightarrow{+y} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -i & i \end{pmatrix} \left(\frac{1}{2}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hat{P}_- |-\rangle \xrightarrow{+y} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -i & i \end{pmatrix} \left(\frac{-i}{2}\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{-i}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{P}_+ |-\rangle \xrightarrow{+y} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \left(\frac{-i}{2}\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hat{P}_+^2 \xrightarrow{+y} \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{bmatrix} \hat{P}_+ \end{bmatrix}^2$$

$$\hat{P}_-^2 \xrightarrow{+y} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{bmatrix} \hat{P}_- \end{bmatrix}^2$$

$$\hat{P}_+ \hat{P}_- \xrightarrow{+y} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_- \hat{P}_+ \xrightarrow{+y} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$