Physics 491 Fall 2015

Recitation #7 Solutions

① $$\left(|R\rangle, |L\rangle \right) = \left(|x\rangle, |y\rangle \right) \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{\hat{S}}$$

$$\hat{S}^{+}\hat{S} = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|\psi\rangle \xrightarrow[x,y]{} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

$$|\psi\rangle \xrightarrow[R,L]{} \hat{S}^{+} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_x - i\psi_y \\ \psi_x + i\psi_y \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|R\rangle (\psi_x - i\psi_y) + |L\rangle (\psi_x + i\psi_y) \right)$$

we can check:

$$\langle x|\psi\rangle = \frac{1}{\sqrt{2}} (\psi_x - i\psi_y) \underbrace{\langle x|R\rangle}_{\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}} (\psi_x + i\psi_y) \underbrace{\langle x|L\rangle}_{\frac{1}{\sqrt{2}}}$$

$$= \psi_x$$

$$\langle y|\psi\rangle = \frac{1}{\sqrt{2}} (\psi_x - i\psi_y) \underbrace{\langle y|R\rangle}_{= i/\sqrt{2}} + \frac{1}{\sqrt{2}} (\psi_x + i\psi_y) \underbrace{\langle y|L\rangle}_{= \frac{-i}{\sqrt{2}}}$$

$$= \psi_y$$

(2) $$\left[\hat{J}_z\right]_{spherical} = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\hat{J}_z\right]_{linear} = \hbar \left(\frac{1}{\sqrt{2}}\right)^2 \overset{S^\dagger}{\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{S}{\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}}_{\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}}$$

$$= \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

transform to get the eigenstate using $\hat{S}$

$$\left(\ \right)_{\substack{linear \\ +\hbar}} = \hat{S} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\left(\ \right)_{\substack{linear \\ -\hbar}} = \hat{S} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

check: $$\hat{J}_z |R\rangle \xrightarrow[linear]{} \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = +\hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\hat{J}_z |L\rangle \xrightarrow[linear]{} \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = -\hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Part 2:

(3) $|R\rangle\langle R| \xrightarrow[\text{sph}]{} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $|L\rangle\langle L| \xrightarrow[\text{sph}]{} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$|R\rangle\langle R| \xrightarrow[\text{linear}]{} \hat{S}^{+} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \hat{S}$$

$$= \left(\tfrac{1}{2}\right) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}_{\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

use a short cut,

$$|L\rangle\langle L| = \hat{I} - |R\rangle\langle R| \xrightarrow[\text{linear}]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & +i \\ -i & 1 \end{pmatrix}$$

part 1: easy to show that operator has correct eigenvalue

$$\hat{J}_z |R\rangle = \hbar\left(|R\rangle\langle R| - |L\rangle\langle L|\right)|R\rangle = +\hbar|R\rangle$$

$$\hat{J}_z |L\rangle = \hbar\left(|R\rangle\langle R| - |L\rangle\langle L|\right)|L\rangle = -\hbar|L\rangle$$

OR, write explicitly in spherical basis:

$$\hbar\left(|R\rangle\langle R| - |L\rangle\langle L|\right) = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$