Phys 491 Fall 2022

Review II

What we covered:

1 1D Q.M. bound states, scattering
2 spin
3 rotation of spinor, change of basis
4 general theory of angular momentum
5 photon
6 unitary transformation
7 two state systems
8 multiparticle states, EPR
9 simple harmonic oscillator
10 FPI (not on final)

Example 1 Particle in box

$$\Psi(x,t) = e^{-iE_n t/\hbar}\,\psi_n(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n + V(x)\,\psi_n = E\,\psi_n$$

$$V(x) = \begin{cases} 0 & |x| < a/2 \\ \infty & |x| > a/2 \end{cases}$$

Energy quantized: wave number $k_n = \frac{n\pi}{a}$

$$E_n = \frac{p_n^2}{2m} = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2}{2ma^2}\,n^2$$

$$\langle x|n\rangle = \psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\cos k_n x & n \text{ odd} \\ \sqrt{\frac{2}{a}}\sin k_n x & n \text{ even} \end{cases}$$

orthonormal $\langle n|m\rangle = \delta_{nm} = \int_{-a/2}^{a/2} \psi_n(x)\,\psi_m(x)\,dx$

complete:
$$\psi(x) = \sum_{n=1}^{\infty} C_n\,\psi_n(x)$$

so $$C_n = \int_{-a/2}^{a/2} \psi_n^*(x)\,\psi(x)\,dx$$

amplitude to be in state ψ_n and to measure energy E_n.

Consider superposition state:

$$\Psi(x) = \frac{3}{5}\psi_1(x) + \frac{4i}{5}\psi_3(x)$$

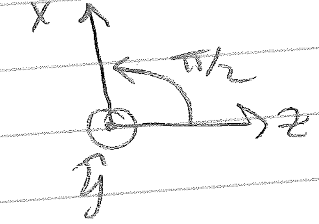
$$= \sqrt{\frac{2}{a}}\left(\frac{3}{5}\right)\cos\frac{\pi x}{a} + \sqrt{\frac{2}{a}}\left(\frac{4i}{5}\right)\cos\left(\frac{3\pi x}{a}\right)$$

probability to measure E_1: use orthogonality

$$C_1 = \int \psi_1(x)\,\Psi(x)\,dx = \frac{3}{5}$$

$$P_1 = |C_1|^2 = \frac{9}{25}$$

Example 2 Spinor Basis



$$\hat{R}^s(\theta\hat{y}) = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ +\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\theta = \pi/2$$

$$\big(|+x\rangle, |-x\rangle\big) = \big(|+z\rangle, |-z\rangle\big)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

or $|\pm x\rangle = \big(\pm|+z\rangle + |-z\rangle\big)/\sqrt{2}$

we chose phase $|-x\rangle = \big(|+z\rangle - |-z\rangle\big)/\sqrt{2}$

corresponding to $\hat{R}' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ row interchange

$[\hat{S}_x]$ in $\hat{z}$ basis

$$[\hat{S}_x]_{ij} = \begin{pmatrix} \langle +z|\hat{S}_x|+z\rangle & \langle +z|\hat{S}_x|-z\rangle \\ \langle -z|\hat{S}_x|+z\rangle & \langle -z|\hat{S}_x|+z\rangle \end{pmatrix}$$

invert change of basis to get

$$|+z\rangle = \frac{1}{\sqrt{2}}\big(|+x\rangle + |-x\rangle\big)$$

$$|-z\rangle = \frac{1}{\sqrt{2}}\big(|+x\rangle - |-x\rangle\big)$$

$$\hat{S}_x|+z\rangle = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}|+x\rangle - \frac{\hbar}{2}|-x\rangle\right) = \frac{\hbar}{2}|-z\rangle$$

$$\hat{S}_x|-z\rangle = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}|+x\rangle - \left(-\frac{\hbar}{2}\right)|-x\rangle\right) = \frac{\hbar}{2}|-z\rangle$$

giving $[S_x]_{ij} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Then for spinor state

$$|\chi\rangle = 3/5\,|+z\rangle + \frac{4i}{5}|-z\rangle \underset{z}{\longrightarrow} \frac{1}{5}\begin{pmatrix}3\\4i\end{pmatrix}$$

expectation value

$$\langle \hat{S}_z \rangle = \frac{1}{5}(3, -4i)\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\frac{1}{5}\begin{pmatrix}3\\4i\end{pmatrix}$$

$$= \left(\frac{1}{5}\right)^2\frac{\hbar}{2}(3, -4i)\begin{pmatrix}3\\-4i\end{pmatrix} = \left(\frac{1}{5}\right)^2\frac{\hbar}{2}(9-16)$$

$$= \frac{-7}{25}\left(\frac{\hbar}{2}\right)$$

$$\langle \hat{S}_x \rangle = \frac{1}{5}(3, -4i)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{5}\begin{pmatrix}3\\4i\end{pmatrix}$$

$$= \left(\frac{1}{5}\right)^2\frac{\hbar}{2}(3, -4i)\begin{pmatrix}4i\\3\end{pmatrix}$$

$$= \left(\frac{1}{5}\right)^2\frac{\hbar}{2}(12i - 12i) = 0$$

we can check amplitude to measure $|+x\rangle$:

$$\langle +x|\chi\rangle = \frac{1}{\sqrt{2}}\left(\langle +z| + \langle -z|\right)\left[\frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\left(\frac{3}{5}\right) + \left(\frac{4i}{5}\right)\right]$$

$$P_{+x} = |\langle +x|\chi\rangle|^2 = \frac{1}{2}\left(\frac{1}{5}\right)^2(3-4i)(3+4i)$$

$$\frac{1}{2}\left(\frac{1}{5}\right)^2(9+16) = \frac{1}{2}$$

$*$ $P_{-x} = \frac{1}{2}$ so $\langle \hat{S}_x \rangle = P_{+x}\left(\frac{\hbar}{2}\right) + P_{-x}\left(-\frac{\hbar}{2}\right) = 0$

Example 3 General Angular momentum

$|\hat{\vec{J}}|^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ Casimir operator commutes with all $\vec{J}$.

$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j\, m\rangle$$
$$\hat{J}_z |j, m\rangle = m\hbar |j, m\rangle$$

$$\hat{J}_\pm |j, m\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\, |j, m\pm 1\rangle$$

$\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$ raising & lowering

So for spinor $s = 1/2$

$$\hat{S}_+ |1/2, -1/2\rangle = \hbar \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}+1)}\, |1/2, +1/2\rangle$$
$$= \hbar \sqrt{3/4 + (\tfrac{1}{2})^2}\, |1/2, +1/2\rangle = \hbar |\tfrac{1}{2}, +1/2\rangle$$

and $\hat{S}_- |1/2, +1/2\rangle = \hbar |1/2, -1/2\rangle$

in z basis:

$$[\hat{S}_+] = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [\hat{S}_-] = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[\hat{S}_+] |1/2, -1/2\rangle \rightarrow \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \hbar |\tfrac{1}{2}, +1/2\rangle$$

then we can get matricies

$$[\hat{S}_x] = ([\hat{S}_+] + [S_-])\frac{1}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[\hat{S}_y] = ([S_+] - [S_-])\frac{1}{2i} = -\frac{i\hbar}{2}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Example 4 photon

$|R\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle \right)$ this is spherical basis

$|L\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle \right)$

this change of basis is NOT a rotation:

$$\left(|R\rangle, |L\rangle\right) = \left(|x\rangle, |y\rangle\right) \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}}_{\hat{S}}$$

similarity transformation: $\hat{S}^{+}\hat{S} = \hat{I}$

$$\hat{J}_z |R\rangle = \hbar |R\rangle \quad ; \quad \hat{J}_z |L\rangle = -\hbar |L\rangle$$

$$\left[\hat{J}_z\right] = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{spherical basis}$$

suppose $|\psi\rangle = \psi_x |x\rangle + \psi_y |y\rangle$

What are components of $|\psi\rangle$ in spherical basis?

Components transform like $\hat{S}^{+}$:

$$|\psi\rangle \underset{\text{spherical}}{\longrightarrow} \hat{S}^{+}\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_x - i\psi_y \\ \psi_x + i\psi_y \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\psi_x - i\psi_y\right)|R\rangle + \frac{1}{\sqrt{2}}\left(\psi_x + i\psi_y\right)|L\rangle$$

check $\langle x|\psi\rangle = \frac{1}{\sqrt{2}}\left(\psi_x - i\psi_y\right)\underbrace{\langle x|R\rangle}_{1/\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\psi_x + i\psi_y\right)\underbrace{\langle x|L\rangle}_{1/\sqrt{2}}$

$$\langle x|\psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 2\,\psi_x = \psi_x$$

$$\langle y|\psi\rangle = \frac{1}{\sqrt{2}}(\psi_x - i\psi_y)\underbrace{\langle y|R\rangle}_{\frac{i}{\sqrt{2}}} + \frac{1}{\sqrt{2}}(\psi_x + i\psi_y)\underbrace{\langle y|L\rangle}_{\frac{-i}{\sqrt{2}}}$$

$$= \psi_y$$

What is $\hat{J}_z$ in linear basis?

$$[\hat{J}_z]_{linear} = \hat{S}^{+}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\hat{S}$$

$$= \hbar\left(\frac{1}{\sqrt{2}}\right)^2\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$$

$$= \hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

<u>Check</u> $\hat{J}_z|R\rangle \xrightarrow[linear]{} \hbar\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \hbar\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$

$$\rightarrow \hbar|R\rangle$$

projection operator

$$|R\rangle\langle R| \xrightarrow[\text{spherical}]{} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|L\rangle\langle L| \xrightarrow[\text{sph}]{} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so

$$\hbar\left(|R\rangle\langle R| - |L\rangle\langle L|\right) = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow J_z$$

Example: Unitary Transformation

time evolution; $\hat{H}$ independent of time

$$\hat{U}(t) = \exp\left\{-\frac{i}{\hbar}\hat{H}t\right\}$$

$\hat{U}^{+}\hat{U} = \hat{I}$ unitary, $\hat{H}^{+} = \hat{H}$ Hermitian

free particle: $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\hat{H}|p\rangle = \frac{\hat{p}^2}{2m}|p\rangle = \frac{p^2}{2m}p$$

at $t=0$, $|\psi(0)\rangle = |p\rangle$; $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}}$

$$|\psi(t)\rangle = e^{-it\frac{\hat{p}^2}{2m\hbar}}|p\rangle = e^{-it\frac{p^2}{2m\hbar}}|p\rangle$$

then $\langle x|\psi(t)\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)}$

plane wave with $E = \frac{p^2}{2m}$

moving in $+x$ direction

translation of $|p\rangle$ basis: $\hat{W}(p_0) \equiv e^{ip_0\hat{x}/\hbar}$

$\hat{W}^{+}\hat{W} = \hat{I}$

from $[\hat{p}, f(\hat{x})] = -i\hbar\frac{\partial}{\partial\hat{x}}f(\hat{x})$

we get $[\hat{p}, \hat{W}(p_0)] = p_0\hat{W}(p_0)$

Then state $\hat{W}(p_0)|p\rangle$ has momentum eigenvalue

$$\hat{p}\left(\hat{W}|p\rangle\right) = \left[\hat{W}\hat{p} + p_0\hat{W}\right]|p\rangle = (p+p_0)\hat{W}|p\rangle$$

so $\hat{W}(p_0)|p\rangle = |p+p_0\rangle$

Consider arbitrary state $|\psi'\rangle = \hat{W}(p_0)|\psi\rangle$

expectation of $\hat{x}$ is unchanged. Use $[\hat{W}, \hat{x}] = 0$
and $\hat{W}^{+}\hat{W} = \hat{I}$

$$\langle\psi'|\hat{x}|\psi'\rangle = \langle\psi|\hat{W}^{+}(p_0)\,\hat{x}\,\hat{W}(p_0)|\psi\rangle$$

$$= \langle\psi|\hat{W}^{+}\hat{W}\hat{x}|\psi\rangle = \langle\psi|\hat{x}|\psi\rangle$$

But expectation value of $\hat{p}$ shifts:

$$\langle\psi'|\hat{p}|\psi'\rangle = \langle\psi|\hat{W}^{+}\hat{p}\hat{W}|\psi\rangle$$

$$= \langle\psi|\hat{W}^{+}(\hat{W}\hat{p} + p_0\hat{W})|\psi\rangle$$

$$= \langle\psi|\hat{p}|\psi\rangle + p_0\underbrace{\langle\psi|\psi\rangle}_{\text{normalized to } 1}$$

Similarly, for $\hat{T}(a) = e^{-ia\hat{p}/\hbar}$

$$[\hat{x}, \hat{T}(a)] = i\hbar \frac{\partial}{\partial p}\hat{T}(a) = a\hat{T}(a)$$

If $|\psi'\rangle = \hat{T}(a)|\psi\rangle$

$$\langle\psi'|\hat{x}|\psi'\rangle = \langle\psi|\hat{T}^{+}\hat{x}\hat{T}|\psi\rangle$$

$$= \langle\psi|\hat{T}^{+}(\hat{T}\hat{x} + a\hat{T})|\psi\rangle$$

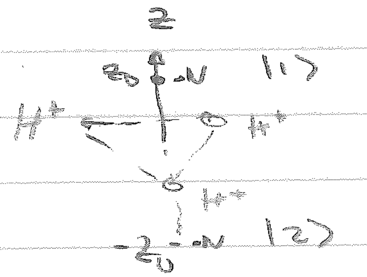
$$= \langle\psi|\hat{x}|\psi\rangle + a\langle\psi|\psi\rangle$$

But $\langle\psi'|\hat{p}|\psi'\rangle = \langle\psi|\hat{p}|\psi\rangle$

Example: NH_3 in static $\vec{E}$ field

$\vec{E} = \mathcal{E}\hat{x}$

ammonia basis



$\hat{H}' = -\vec{\mu}_e \cdot \vec{E}$

states $|1\rangle, |2\rangle$ are eigenstates of electric dipole operator $\vec{\mu}_e$: ($\mu \equiv$ magnitude of dipole)

$$\vec{\mu}_e |1\rangle = -\mu \hat{z} |1\rangle$$
$$\vec{\mu}_e |2\rangle = +\mu \hat{z} |2\rangle$$

$$[\hat{H}]_{1/2} = \begin{bmatrix} E_0 + \mu\mathcal{E} & -A \\ -A & E_0 - \mu\mathcal{E} \end{bmatrix}$$

define $B_\pm \equiv E_0 \pm \mu\mathcal{E}$

$$\begin{bmatrix} B_+ & -A \\ -A & B_- \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = E \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\det \begin{bmatrix} B_+ - E & -A \\ -A & B_- - E \end{bmatrix} = 0$$

$$(B_+ - E)(B_- - E) - A^2 = 0$$

$$B_+ B_- - E B_- - E B_+ + E^2 - A^2 = 0$$

$$B_+ B_- = E_0^2 + (\mu \mathcal{E})^2$$
$$B_+ + B_- = 2 E_0$$

$$E^2 - 2 E_0 E + E_0^2 + (\mu \mathcal{E})^2 - A^2 = 0$$

gives
$$E_\pm = E_0 \pm \sqrt{E_0^2 - E_0^2 - (\mu \mathcal{E})^2 + A^2}$$
$$= E_0 \pm \sqrt{(\mu \mathcal{E})^2 + A^2}$$

Weak field $\mu \mathcal{E} \ll A$

$$E_\pm^{weak} \cong E_0 \pm A \left[1 + \frac{1}{2} \left(\frac{\mu \mathcal{E}}{A} \right)^2 \right]$$ increases quadratically with $\mathcal{E}$

strong field $\mu \mathcal{E} \gg A$

$$E_\pm^{strong} = E_0 \pm \mu \mathcal{E} \left[1 + \frac{1}{2} \left(\frac{A}{\mu \mathcal{E}} \right)^2 \right]$$

$$\simeq E_0 \pm \mu \mathcal{E}$$ increase linearly

<u>We found induced electric dipole moment in weak field.</u>

$|I\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |II\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\left[\hat{\vec{\mu}}_e\right]_{1,2} = \mu \hat{z} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

so $\langle I | \hat{\vec{\mu}}_e | I \rangle = 0 = \langle II | \hat{\mu}_e | II \rangle$

<u>But in weak field eigenstates are modified</u>

$$|I'\rangle = \frac{1}{N^{1/2}} \left[A|1\rangle + \left(\mu\mathcal{E} + A + \frac{1}{2}\frac{(\mu\mathcal{E})^2}{A}\right)|2\rangle \right]$$

with $N \approx 2A^2 + \mu\mathcal{E}A \approx 2A^2$

$$|I'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 + \frac{\mu\mathcal{E}}{A} + \frac{1}{2}\frac{(\mu\mathcal{E})^2}{A^2} \end{pmatrix} \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 + \frac{\mu\mathcal{E}}{A} \end{pmatrix}$$

$$\langle I' | \hat{\vec{\mu}}_e | I' \rangle = \frac{1}{2}\mu\hat{z} \left(1, 1 + \frac{\mu\mathcal{E}}{A}\right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 + \frac{\mu\mathcal{E}}{A} \end{pmatrix}$$

$$= \frac{1}{2}\mu\hat{z} \left(1, 1 + \frac{\mu\mathcal{E}}{A}\right) \begin{pmatrix} -1 \\ 1 + \frac{\mu\mathcal{E}}{A} \end{pmatrix}$$

$$= \frac{1}{2}\mu\hat{z} \left(-1 + 1 + 2\frac{\mu\mathcal{E}}{A}\right) \quad \text{dropping } \left(\frac{\mu\mathcal{E}}{A}\right)^2 \text{ term}$$

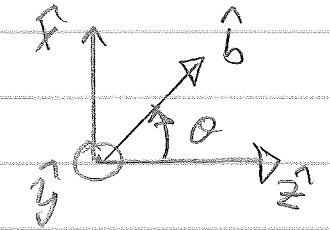
$$= \hat{z}\,\mu \left(\frac{\mu\mathcal{E}}{A}\right) \quad \underline{\text{linear in } \mathcal{E}}$$

Multiparticle state Example

State with $\hat{S}^2|0,0\rangle = 0$, $\hat{S}_z|0,0\rangle = 0$
for total $\vec{S} = \vec{S}_1 + \vec{S}_2$ is

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left[|+z\rangle_1|-z\rangle_2 - |-z\rangle_1|+z\rangle_2\right]$$

Invariant under rotation. Consider



Change of basis from $SG_{\hat{b}}$ eigenstate to $SG_{\hat{z}}$ eigenstate is spinor rotation

$$\hat{R}^{S\dagger}(\theta\hat{y}) = \hat{R}^S(-\theta\hat{y}) = \begin{pmatrix} \cos\theta/_2 & \sin\theta/_2 \\ -\sin\theta/_2 & \cos\theta/_2 \end{pmatrix}$$

$$\left(|+z\rangle, |-z\rangle\right) = \left(|+b\rangle, |+b\rangle\right)\begin{pmatrix} c & s \\ -s & c \end{pmatrix}_{\theta/2}$$ suppress arg. of cos, sin

$|+z\rangle = c|+b\rangle - s|-b\rangle$
$|+z\rangle = s|+b\rangle + c|-b\rangle$

Then $|0,0\rangle = \frac{1}{\sqrt{2}}\left(c|+b\rangle_1 - s|-b\rangle_1\right)\otimes\left(s|+b\rangle_2 + c|-b\rangle_2\right)$

$$-\frac{1}{\sqrt{2}}\left(s|+b\rangle_1 + c|-b\rangle_1\right)\otimes\left(c|+b\rangle_2 - s|-b\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}\Big[\underline{cs\,|{+b}\rangle_1\,|{+b}\rangle_2} + c^2\,|{+b}\rangle_1|{-b}\rangle_2$$

$$-s^2\,|{-b}\rangle_1|{+b}\rangle_2 - \underline{cs\,|{-b}\rangle_1\,|{-b}\rangle_2}$$

$$-\underline{sc\,|{+b}\rangle_1\,|{+b}\rangle_2} + s^2\,|{+b}\rangle_1|{-b}\rangle_2$$

$$-c^2\,|{-b}\rangle_1|{+b}\rangle_2 + \underline{cs\,|{-b}\rangle_1\,|{-b}\rangle_2}\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[(c^2+s^2)\,|{+b}\rangle_1|{-b}\rangle_2 - (c^2+s^2)\,|{-b}\rangle_1|{+b}\rangle_2\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[|{+b}\rangle_1|{-b}\rangle_2 - |{-b}\rangle_1|{+b}\rangle_2\Big]$$

state $|0,0\rangle$ is invariant, same in any basis

Simple Harmonic Oscillator Example

$$\hat{q} \equiv \frac{\hat{p}}{\sqrt{\hbar m \omega}} \quad ; \quad \hat{y} \equiv \hat{x}\sqrt{\frac{m\omega}{\hbar}}$$

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{q}^2 + \hat{y}^2\right)$$

$$\left[\hat{y}, \hat{q}\right] = i$$

raising $\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\hat{y} - i\hat{q}\right)$ $\qquad \left[\hat{a}, \hat{a}^{+}\right] = 1$

lowering $\hat{a} = \frac{1}{\sqrt{2}}\left(y + i\hat{q}\right)$

$$\hat{H} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right) \qquad \hat{n} \equiv \hat{a}^{+}\hat{a} \text{ number operator}$$

$$\hat{a}^{+}|n\rangle = \sqrt{(n+1)}\,|n+1\rangle \; ; \; \hat{a}^{-}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{H}|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)|n\rangle \text{ eigenstates}$$

$$\hat{y} = \frac{1}{\sqrt{2}}\left(\hat{a}^{+} + \hat{a}\right)$$

$$\langle n|\hat{y}|n\rangle = \frac{1}{\sqrt{2}}\langle n|\left(\hat{a}^{+} + \hat{a}\right)|n\rangle = 0$$

$$\langle n|\hat{y}^2|n\rangle = \frac{1}{2}\langle n|\left(\hat{a}^{+2} + \hat{a}^2 + \underbrace{\hat{a}^{+}a + aa^{+}}_{2\hat{n}+1}\right)|n\rangle$$

$$= \frac{1}{2}(2n+1)\langle n|n\rangle = n + \tfrac{1}{2}$$

$\Delta y = \sqrt{n + \frac{1}{2}}$ $\qquad \Delta y \Delta q = n + \frac{1}{2}$

similarly $\Delta q = \sqrt{n + \frac{1}{2}}$ $\qquad \Delta x \Delta p = \hbar\left(n + \frac{1}{2}\right)$

Time evolution of $\langle x\rangle$, $\langle p\rangle$

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle$$

$$[\hat{H},\hat{x}] = \hbar\omega\left[\hat{a}^{+}a+\tfrac{1}{2},\ \sqrt{\tfrac{\hbar}{2m\omega}}(\hat{a}^{+}+\hat{a})\right]$$

$$= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left[\hat{a}^{+}\hat{a},\ \hat{a}^{+}+\hat{a}\right]$$

$$= \hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left\{\underbrace{[\hat{a}^{+}\hat{a},\hat{a}^{+}]}_{\hat{a}^{+}}+\underbrace{[\hat{a}^{+}\hat{a},\hat{a}]}_{-\hat{a}}\right\}$$

$$= \frac{\hbar}{im}\hat{p}$$

So $\dfrac{d}{dt}\langle\hat{x}\rangle = \dfrac{1}{m}\langle\hat{p}\rangle$

note: for energy eigenstate $\langle n|\hat{x}|n\rangle = 0 = \langle n|\hat{p}|n\rangle$
but not for <u>superposition state</u>.