Nov. 22 2021

Lecture: Review

Topics: 1) single particle spin ½

2) single massive particle (spin 1)

3) Time evolution of single particle spin state

4) Two particle states

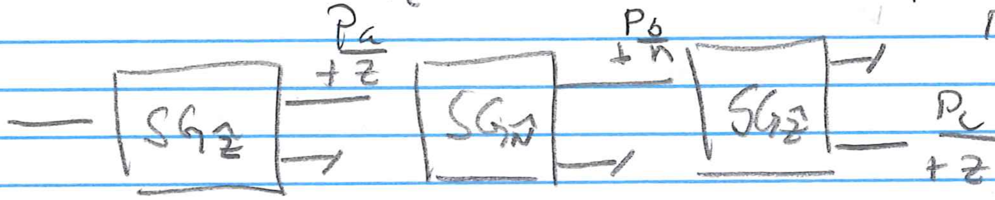
Example: Stern Gerlach

initial state $|+y\rangle$

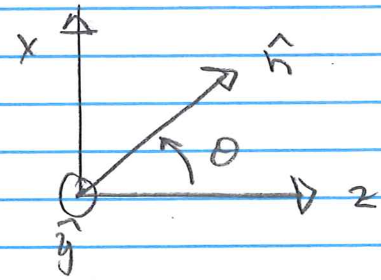
Series of measurements:

measure SG_z select $+z$
measure SG_n select $+n$
measure SG_z select $-z$

probabilities P_a, P_b, P_c



$\hat{n}\cdot\hat{z} = \cos\theta$ where $\hat{n}\cdot\hat{y} = 0$



$$R^s(\theta \hat{y}) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_{\theta/2}$$

$$\left(|+n\rangle, |-n\rangle\right) = \left(|+z\rangle, |-z\rangle\right)\begin{pmatrix} c & -s \\ s & c \end{pmatrix}_{\theta/2}$$

$$|+n\rangle = c|+z\rangle + s|-z\rangle$$
$$|-n\rangle = -s|+z\rangle + c|-z\rangle$$

In $\hat{z}$ basis: $|+z\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$; $|-z\rangle \to \begin{pmatrix}0\\1\end{pmatrix}$

$|+y\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$$P_a = |\langle +z|+y\rangle|^2 = \left|(1,0), \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$P_b = |\langle +n|+z\rangle|^2 = \cos^2 \theta/2$$

$$P_c = |\langle -z|+n\rangle|^2 = \sin^2 \theta/2$$

$$P = P_a \cdot P_b \cdot P_c = \frac{1}{2}\cos^2\theta/2 \; \sin^2\theta/2 = \frac{1}{8}\sin^2\theta$$

for $\theta = 0$ or π $\quad P = 0$ since $\langle +z|-z\rangle = 0$

for $\theta = \pi/2$ $\quad P = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

<u>Example</u> massive spin-1

$$|\psi\rangle \xrightarrow[z \text{ basis}]{} N \begin{pmatrix}1\\2\\3i\end{pmatrix}$$

normalize $\quad N^2 (1, 2, -3i)\begin{pmatrix}1\\2\\3i\end{pmatrix} = N^2 \cdot 14 = 1$

$$N = 1/\sqrt{14}$$

Spin $\hat{S}_z$ eigenvalues are $1, 0, -1$ probabilities

$$P_{+1} = \frac{1}{14}; \; P_0 = \frac{4}{14}; \; P_{-1} = \frac{9}{14}$$

$$[\hat{S}_z] = \hbar \, \mathrm{diag}(1, 0, -1)$$

$$\langle S_z \rangle = \frac{\hbar}{14}(1, 2, -3i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} = \frac{-8\hbar}{14} = \frac{-4}{7}\hbar$$

$$\langle \hat{S}_x \rangle = \frac{\hbar}{14}(1, 2, -3i)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}\,14}(1, 2, -3i)\begin{pmatrix} 2 \\ 1+3i \\ 2 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}\,14}\left[2 + 2(1 + \cancel{3i}) - \cancel{(3i)}2\right] = \frac{+2}{7\sqrt{2}}\hbar$$

<u>example</u> Spin-1 particle with charge q, g-factor g, mass m in magnetic field $\vec{B}$.

Hamiltonian $\quad \hat{H} = -\vec{\mu}\cdot\vec{B} = -\frac{gqB}{2m}\hat{S}_z \equiv -\omega_0 \hat{S}_z$

take $\vec{B} = B\hat{z}$ $\quad \vec{\mu} = \frac{gq}{2m}\vec{S}$

let

$$|\psi(0)\rangle = \frac{1}{2}\left[|1,1\rangle + i\sqrt{2}\,|1,0\rangle - |1,-1\rangle\right]$$

$$\xrightarrow[\text{z basis}]{} \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

$$\hat{U}(t) = e^{-i\frac{\hat{H}}{\hbar}t} = e^{+i\frac{\omega_0}{\hbar}\hat{S}_z}$$

$$|\psi(t)\rangle = \frac{1}{2}\begin{pmatrix} e^{i\omega_0 t} \\ i\sqrt{2} \\ -e^{-i\omega_0 t}\end{pmatrix}$$

$$\langle \hat{S}_z(t)\rangle = \frac{\hbar}{4}\left(e^{-i\omega_0 t}, -i\sqrt{2}, -e^{+i\omega_0 t}\right)\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix} e^{i\omega_0 t}\\ i\sqrt{2} \\ e^{-i\omega_0 t}\end{pmatrix}$$

$$= 0$$

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$$

$$\langle \hat{S}_x(t)\rangle = \frac{\hbar}{\sqrt{2}}\left(\frac{1}{4}\right)\left(e^{-i\omega_0 t}, -i\sqrt{2}, -e^{+i\omega_0 t}\right)\underbrace{\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix} e^{i\omega_0 t}\\ i\sqrt{2} \\ e^{-i\omega_0 t}\end{pmatrix}}_{\begin{pmatrix} i\sqrt{2} \\ e^{i\omega_0 t} - e^{-i\omega_0 t} \\ i\sqrt{2}\end{pmatrix}}$$

$$= \frac{\hbar}{\sqrt{2}}\left(\frac{1}{4}\right)\left[i\sqrt{2}e^{-i\omega_0 t} - i\sqrt{2}e^{i\omega_0 t} - i\sqrt{2}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)\right]$$

$$= \frac{\hbar}{4}\cdot 2\left(-ie^{i\omega_0 t} + ie^{-i\omega_0 t}\right) = \underline{\hbar \sin\omega_0 t}$$

Classically, spin precesses about $\hat{z}$ axis

Classical Precession

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \frac{d\vec{S}}{dt} = \vec{\omega} \times \vec{S}$$

at $t=0$ $\quad \langle \hat{S}_x \rangle = 0$ & $\langle \hat{S}_z \rangle = 0$

$$[\hat{S}_y] = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

at $t=0$

$$\langle S_y \rangle = \frac{\hbar}{4\sqrt{2}} \begin{pmatrix} 1 & -i\sqrt{2} & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}}$$

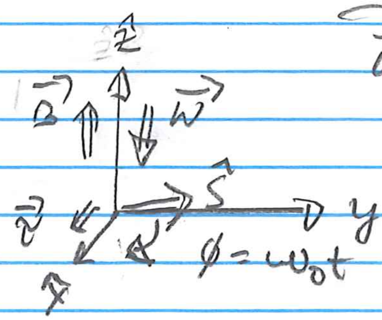
$$\frac{\hbar}{4\sqrt{2}} \begin{pmatrix} 1, & -i\sqrt{2}, & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 2i \\ -\sqrt{2} \end{pmatrix}$$

$$= \frac{\hbar}{4\sqrt{2}} \left(\sqrt{2} + 2\sqrt{2} + \sqrt{2} \right) = +\hbar$$

So at $t=0$ classical $\vec{\mu} = \frac{gq}{2m}(\hbar \hat{y})$

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \underbrace{\left(\frac{egB}{2m}\right)}_{\omega_0} \hbar \left(\hat{y} \times \hat{z}\right) = \hbar \omega_0 \hat{x}$$

$$\frac{d\vec{S}}{dt} = \vec{\omega} \times \vec{S} = \underbrace{\left(-\omega_0 \hat{z}\right)}_{\vec{\omega}} \times \left(\hbar \hat{y}\right) = \hbar \omega_0 \hat{x}$$



consistent with Q.M.

$\langle \hat{S}_x \rangle = \hbar \sin(\omega_0 t)$

Example Spin-1/2 particle in $\vec{B}$ field

$\vec{B} = B\hat{x}$ $\qquad |\psi(0)\rangle = |+z\rangle$

$$\hat{H} = \omega_0 \hat{S}_x = \frac{\hbar\omega_0}{2}\hat{\sigma}_x$$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\frac{\omega_0 t}{2}\hat{\sigma}_x}$$

with $\hat{\sigma}_x^2 = \hat{I}$

$$\hat{U}(t) = I - i\frac{\omega_0 t}{2}\sigma_x - \left(\frac{\omega_0 t}{2}\right)^2\frac{\sigma_x^2}{2!} - i\left(\frac{\omega_0 t}{2}\right)^3\frac{\sigma_x^3}{3!}$$

$$= \hat{I}\cos\left(\frac{\omega_0 t}{2}\right) - i\sigma_x \sin\frac{\omega_0 t}{2}$$

$$= \begin{pmatrix} c & -is \\ -is & c \end{pmatrix}_{\theta = \frac{\omega_0 t}{2}}$$

$$|\psi(t)\rangle \xrightarrow[z\ \text{basis}]{} \begin{pmatrix} \cos\omega_0 t/2 \\ -i\sin\frac{\omega_0 t}{2} \end{pmatrix}$$

$$\langle\hat{S}_z(t)\rangle = \frac{\hbar}{2}(c, +is)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c \\ -is \end{pmatrix} = \frac{\hbar}{2}(c^2 - s^2)$$

$$= \frac{\hbar}{2}\cos\omega_0 t$$

Example: 2 spin ½ particles with

$$\hat{H} = \frac{2A}{\hbar^2}\left(\hat{S}^1_x \hat{S}^2_x + \hat{S}^1_y \hat{S}^2_y\right)$$

1) $\hat{S}_z = \hat{S}_z^1 + \hat{S}_z^2$ is conserved

$$[\hat{S}_z, \hat{H}] = [\hat{S}^1_z, \hat{H}] + [\hat{S}^2_z, \hat{H}]$$

$$[S^1_z, S^1_x]\hat{S}^2_x + [\hat{S}^1_z, \hat{S}^1_y]\hat{S}^2_y$$

$$+ \hat{S}^1_x[\hat{S}^2_z, \hat{S}^2_x] + \hat{S}^1_y[S^2_z, S^2_y]$$

$$= i\hbar \hat{S}^1_y \hat{S}^2_x - i\hbar \hat{S}^1_x \hat{S}^2_y$$

$$+ i\hbar \hat{S}^1_x \hat{S}^2_y - i\hbar \hat{S}^1_y \hat{S}^2_x = 0$$

2) $\hat{H}$ in basis $|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle$

$$\hat{H} = \frac{2A}{\hbar^2}\left[\hat{\vec{S}}^1 \cdot \hat{\vec{S}}^2 - \hat{S}^1_z \hat{S}^2_z\right]$$

$$\hat{S}^2 = 2\hat{\vec{S}}^1 \cdot \hat{\vec{S}}^2 + (\hat{S}^1)^2 + (\hat{S}^2)^2$$

$$\hat{H} = \frac{A}{\hbar^2}\left[\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2 - 2\hat{S}^1_z \hat{S}^2_z\right]$$

$$\hat{S}^2|1,m\rangle = \hbar^2 1(1+1)|1,m\rangle = 2\hbar^2|1,m\rangle$$

$$(\hat{S}^1)^2|1,m\rangle = \frac{\hbar^2}{4}|1,m\rangle = (\hat{S}_2)^2|1,m\rangle$$

$$\frac{2A}{\hbar^2}\left[\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right] = A\,\text{diag}\left(2-\tfrac{3}{2}, 2-\tfrac{3}{2}, 2-\tfrac{3}{2}, -\tfrac{3}{2}\right)$$

$$= \frac{A}{2}\,\text{diag}(1,1,1,-3)$$

$$\frac{2A}{\hbar^2}\left[\hat{S}_z\cdot\hat{S}_z\right] = 2A\,\mathrm{diag}\left(+\frac{1}{4}, -\frac{1}{4}, +\frac{1}{4}, -\frac{1}{4}\right)$$

$$= \frac{A}{2}\,\mathrm{diag}(1,-1,1,-1)$$

$$\boxed{[\hat{H}] = \frac{A}{2}\,\mathrm{diag}\,(0,2,0,-2)}$$

$\hat{H}$, $\hat{S}^2$ simultaneously diagonalized

so $[\hat{H}, \hat{S}^2] = 0$

3) Is $\hat{H}$ rotationally invariant?

manifestly not.

eigenvalue $|1,m\rangle$ not all the same

$[\hat{H}, \hat{S}_x] \neq 0$ $\quad [\hat{H}, \hat{S}_y] \neq 0$

For Rotational invariance, $[\hat{R}^s, \hat{H}] = 0$

$$\hat{R}^s = \exp\left(-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}\right),$$

therefore $[\hat{H}, \hat{S}_i] = 0$ $_{i=x,y,z}$ & $[\hat{H}, \hat{S}^2] = 0$

Example (Landau & Lifshitz)

Spin-1/2 operator $\vec{S}$ consider rotationally invariant operator

$$\hat{A} = a\,\underline{I} + 2\vec{b}\cdot\left(\frac{\hat{\vec{S}}}{\hbar}\right) \qquad \begin{array}{l} a \text{ real const.} \\ \vec{b} \text{ real vector} \end{array}$$

for arbitrary function $f(\hat{A})$, show

$$(*) \quad f(\hat{A}) = \alpha + 2\beta\left(\frac{\vec{b}}{b}\right)\cdot\left(\frac{\vec{S}}{\hbar}\right) \qquad \text{find } \alpha, \beta$$

Use $\hat{b}$ basis $\quad 2\left(\frac{\vec{b}\cdot\vec{S}}{\hbar}\right)\left|\tfrac{1}{2}, m\right\rangle_b = 2b\,m\left|\tfrac{1}{2}, m\right\rangle_b$

by Taylor expansion of $f(\hat{A})$

$$f(\hat{A})\left|\tfrac{1}{2}, m\right\rangle = f(a + 2bm)\left|\tfrac{1}{2}, m\right\rangle \qquad m = \pm\tfrac{1}{2}$$

or $\quad f(\hat{A})\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = f(a \pm b)\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$

Eigenvalues are:

$$f(\hat{A})\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = \left(\alpha + \beta\left(\frac{\vec{b}}{b}\right)\cdot\left(\frac{\vec{S}}{\hbar}\right)\right)\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

$$= (\alpha \pm \beta)\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

Comparing, we find:

$$\alpha = \tfrac{1}{2}\left[f(a+b) + f(a-b)\right]$$

$$\beta = \tfrac{1}{2}\left[f(a+b) - f(a-b)\right]$$

By rotational invariance, holds in any basis so (*) is operator identity