HW# 5-Solutions

11.3
$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\hat{H}_1 = \frac{1}{2} m \omega_1^2 \hat{x}^2 \qquad \omega_1 \ll \omega$$

problem is trivially solved exactly:

$$E_N = \hbar\sqrt{\omega^2 + \omega_1^2}\left(N + \frac{1}{2}\right) \qquad N = 0, 1, \ldots$$

In perturbation theory:

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{a} + \hat{a}^+)^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^2 + \hat{a}^{+2} + 2\hat{a}^+\hat{a} + 1\right)$$

1st order –

$$E_N^{(1)} = \frac{1}{2} m\omega_1^2 \left(\frac{\hbar}{2m\omega}\right) \langle N | (2\hat{a}^+\hat{a} + 1) | N\rangle$$

$$= \frac{1}{2}\hbar\omega \left(\frac{\omega_1}{\omega}\right)^2 \left(N + \frac{1}{2}\right)$$

2nd order –

$$E_N^{(2)} = \sum \frac{|\langle k|\hat{H}'|N\rangle|^2}{E_N^0 - E_k^0} = \frac{|\langle N-2|\hat{H}'|N\rangle|^2}{2\hbar\omega} - \frac{|\langle N+2|\hat{H}'|N\rangle|^2}{2\hbar\omega}$$

$$\langle N-2|H'|N\rangle = \frac{1}{2} m\omega_1^2 \left(\frac{\hbar}{2m\omega}\right) \underbrace{\langle N-2|\hat{a}^2|N\rangle}_{\sqrt{N(N-1)}}$$

$$\langle N+2|H'|N\rangle = \frac{1}{2} m\omega_1^2 \left(\frac{\hbar}{2m\omega}\right) \underbrace{\langle N+2|\hat{a}^{+2}|N\rangle}_{\sqrt{(N+1)(N+2)}}$$

$$E_N^{(2)} = \left(\frac{1}{4}\right)^2 (\hbar\omega_1)^2 \left(\frac{\omega_1}{\omega}\right)^2 \frac{1}{2\hbar\omega}\Big[\underbrace{n(N-1) - (N+1)(N+2)}_{-4\left(N+\frac{1}{2}\right)}\Big]$$

$$= -\frac{1}{8}\hbar\omega\left(\frac{\omega_1}{\omega}\right)^4\left(N+\frac{1}{2}\right)$$

thus the energy corrected to second order is

$$E_N = \hbar\omega\left(N+\frac{1}{2}\right)\left\{1+\frac{1}{2}\left(\frac{\omega_1}{\omega}\right)^2-\frac{1}{8}\left(\frac{\omega_1}{\omega}\right)^4\right\}$$

which agrees with the expansion of $\sqrt{1+\frac{\omega_1}{\omega}}$.

11.4 perturbation of a particle in a 1D box.

$$\langle x|1\rangle = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} \qquad \text{ground state}$$

$$E_1^{(0)} = \frac{\pi^2 \hbar^2}{2mL^2}$$

(a) add constant perturbation $\hat{H}' = V_1$
We would expect that the energy would increase by the constant V_1 and indeed:

$$E_1^{(1)} = \langle 1|\hat{H}'|1\rangle = V_1 \langle 1|1\rangle = V_1$$

(b) linear perturbation $\hat{H}' = \frac{\varepsilon E_1^{(0)}}{L} \hat{x}$

$$E_1^{(1)} = \frac{\varepsilon E_1^{(0)}}{L} \langle 1|\hat{x}|1\rangle$$

$$= \frac{\varepsilon E_1^{(0)}}{L} \left(\frac{2}{L}\right) \int_0^L \sin^2\left(\frac{\pi x}{L}\right) x\,dx$$

$$= \frac{\varepsilon E_1^{(0)}}{L} \left(\frac{2}{L}\right)\left(\frac{L}{\pi^2}\right) \int_0^\pi du\, u \sin^2 u$$

$$= \frac{\varepsilon E_1^{(0)}}{L} \left(\frac{2}{L}\right)\left(\frac{L}{\pi^2}\right) \frac{\pi^2}{4} = \frac{\varepsilon}{2} E_1^{(0)}$$

$$E_1 \cong E_1^{(0)}\left(1 + \frac{\varepsilon}{2}\right)$$

11.6 $\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} = -\mu_0 \vec{\sigma} \cdot \vec{B}$

$$\mu_0 = \frac{gq\hbar}{4mc}$$

take $\vec{B} = B_0 \hat{k} + B_2 \hat{j}$, the exact eigenstates are simply the states aligned & antialigned with $\vec{B}$ with eigenvalues,

$$\hat{H} |\pm B\rangle = \mp \mu_0 |\vec{B}| \, |\pm B\rangle \equiv E_\pm |\pm B\rangle$$

$$\text{with } |\vec{B}| = \sqrt{B_0^2 + B_2^2} \approx B_0 \left(1 + \frac{1}{2}\left(\frac{B_2}{B_0}\right)^2\right)$$

Consider $B_2 \hat{j}$ as a perturbation:

$$\hat{H}^0 |\pm z\rangle = \mp \mu_0 B_0 |\pm z\rangle \equiv E_\pm^0 |\pm z\rangle$$

$$\text{and } \hat{H}' = -(\vec{\mu} \cdot \hat{j}) B_2 = -\mu_0 B_2 \hat{\sigma}_y$$

$$\text{recall } \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ in the } |\pm z\rangle \text{ basis}$$

the first order corrections vanish -

$$E_\pm^{(1)} = -\mu_0 B_2 \langle \pm | \hat{\sigma}_y | \pm \rangle = 0$$

The second order corrections are,

$$E_-^{(2)} = \frac{|\langle + | \hat{H}' | - \rangle|^2}{E_-^0 - E_+^0} = \frac{(\mu_0 B_2)^2}{2\mu_0 B_0}$$

$$E_+^{(2)} = \frac{|\langle - | \hat{H}' | + \rangle|^2}{E_+^0 - E_-^0} = -\frac{(\mu_0 B_2)^2}{2\mu_0 B_0}$$

thus, $E_\pm \cong E_\pm^0 + E_\pm^{(2)} = \mp \mu_0 B_0 \left(1 + \frac{1}{2}\left(\frac{B_2}{B_0}\right)^2\right)$

11.7 (a) We use perturbation theory to calculate the effect of the finite nuclear size.
For a sphere of uniform charge density

$$\rho = \frac{q}{\frac{4}{3}\pi R^3}$$ Gauss's law gives for $r<R$:

$$\int \vec{E}\cdot d\vec{a} = 4\pi \int \rho\, dV$$

$$E(r) = q\frac{r}{R^3}$$

the corresponding electrostatic potential:

$$\Phi(r) = -\frac{q}{2}\frac{r^2}{R^3} + C \qquad r>R$$

Continuity with $\Phi = \frac{q}{r}$ for $r>R$ gives at $r=R$

$$-\frac{q}{2}\frac{R^2}{R^3} + C = \frac{q}{R}$$

$$C = \frac{3}{2}\frac{q}{R}$$

thus $$\Phi(r) = \begin{cases} \frac{3}{2}\frac{q}{R^3}\left(R^2 - \frac{r^2}{3}\right) & r<R \\ \frac{q}{r} & r>R \end{cases}$$

thus the potential energy between a charge $-e$ and $q=e$ is

$$V(r) = \begin{cases} -\frac{3}{2}\frac{e^2}{R^3}\left(R^2 - \frac{r^2}{3}\right) & r<R \\ -\frac{e^2}{r} & r>R \end{cases}$$

$$V' = \begin{cases} \frac{e^2}{r} - \frac{3}{2}\frac{e^2}{R^3}\left(R^2 - \frac{r^2}{3}\right) & r < R \\ 0 & r > R \end{cases}$$

$$E_{1s}^{(1)} = \langle 1s|V'(r)|1s\rangle = \int_0^R r^2 dr\, R_{10}^2 V'(r)$$

$$= a_0^3 \int_0^{\varepsilon} x^2 dx\, R_{10}^2(x) V'(x) \qquad x \equiv \frac{r}{a_0},\ \varepsilon \equiv \frac{R}{a_0}$$

$$V'(x) = \frac{e^2}{a_0}\frac{1}{x} - \frac{3}{2}\frac{e^2}{\varepsilon a_0} + \frac{1}{2}\frac{e^2}{(\varepsilon a_0)^3}(a_0 x)^2$$

$$= \frac{e^2}{a_0}\left[\frac{1}{x} - \frac{3}{2\varepsilon} + \frac{1}{2\varepsilon^3}x^2\right]$$

$$R_{10}^2(x) = \frac{4}{a_0^3} e^{-2x} \approx \frac{4}{a_0^3} \quad \text{since } \varepsilon \ll 1$$

then $$E_{1s}^{(1)} = \frac{e^2}{a_0}(4)\left\{\int_0^{\varepsilon} x\,dx - \frac{3}{2\varepsilon}\int_0^{\varepsilon} x^2 dx + \frac{1}{2\varepsilon^3}\int_0^{\varepsilon} x^4 dx\right\}$$

$$= \frac{4e^2}{a_0}\left\{\frac{1}{2}\varepsilon^2 - \frac{1}{2}\varepsilon^2 + \frac{1}{2\varepsilon^3}\frac{1}{5}\varepsilon^5\right\}$$

$$= \frac{2}{5}\frac{e^2}{a_0}\varepsilon^2$$

$$E_{21}^{(1)} = \langle 2p|V'(r)|2p\rangle = a_0^3\int_0^{\varepsilon} x^2 dx\, R_{21}^2(x) V'(x)$$

$$R_{21}^2(x) = \frac{1}{3}\left(\frac{1}{2a_0}\right)^3 x^2 e^{-2x} \approx \frac{1}{24}\left(\frac{1}{a_0}\right)^3 x^2$$

$$E_{21}^{(1)} = \frac{1}{24}\frac{e^2}{a_0}\left\{\int_0^{\varepsilon} x^3 dx - \frac{3}{2\varepsilon}\int x^4 dx + \frac{1}{2\varepsilon^3}\int_0^{\varepsilon} x^6 dx\right\}$$

$$= \text{const}\,\frac{e^2}{a_0}\,\varepsilon^4 .$$

Thus, the correction to $E_{21}^{(1)}$ is negligible.

The shift in the Lyman α frequency is:

$$\frac{\Delta\nu}{\nu} = \frac{\Delta E^{(1)}(2p\text{-}1s)}{\Delta E^0} = \frac{-\frac{2}{5}\frac{e^2}{a_0}\varepsilon^2}{\frac{1}{2}\frac{e^2}{a_0}\left(1-\frac{1}{2^2}\right)} = -\varepsilon^2\frac{4}{5}\left(\frac{4}{3}\right)$$

$$\varepsilon = 10^{-15}\,m/10^{-10}\,m = 10^{-5} \Rightarrow \frac{\Delta\nu}{\nu} = -10^{-10}$$

this shift is of order $\alpha^4 = \left(\frac{1}{137}\right)^4 = 30\times10^{-10}$

$$\Delta\nu = \frac{10^{-10}(10.2\,eV)}{2\pi\hbar} = \frac{10^{-9}\,eV}{2\pi(6.58)\times10^{-16}\,eV\cdot s} = \frac{10^7}{41}\,Hz = 0.25\,MHz$$

This is small compared to the Lamb shift of 1057 MHz, but due to the present experimental precision must be taken into account!

$$\boxed{\Delta_{exp}(2p\text{-}1s) = 1057.845 \pm 0.009}$$

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Physics 452 HW #5 Solutions

Problem 10-10 spin 1 particle with

$$\hat{H}^0 = a\left(\frac{\hat{S}_z}{\hbar}\right)^2 \quad \& \quad \hat{H}' = b\left[\left(\frac{\hat{S}_x}{\hbar}\right)^2 - \left(\frac{\hat{S}_y}{\hbar}\right)^2\right]$$

$$\hat{H}^0 |1,m\rangle = a m^2 |1,m\rangle$$

states $|1,\pm1\rangle$ are degenerate.

in basis $|1,1\rangle, |1,-1\rangle$

$$[\hat{H}^0] = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

note $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y \Rightarrow \hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$

$$\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$$

so $\hat{S}_x^2 - \hat{S}_y^2 = \frac{1}{2}(\hat{S}_+^2 + \hat{S}_-^2)$

$$\hat{S}_\pm |1,m\rangle = \sqrt{2-(m\pm1)}\,|1,m\pm1\rangle$$

$$\hat{S}_+^2 |1,-1\rangle = 2|1,1\rangle \;;\; \hat{S}_-^2|1,1\rangle = 2|1,-1\rangle$$

$$[\hat{H}'] = b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

diagonalize: $b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^{(1)}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

set determinant to zero gives

$$E^{(1)} = \pm b$$

The exact energies are easily obtained from diagonalizing $[\hat{H}] = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ gives

$$E_\pm = a \pm b$$

11.11 The unperturbed, 2-D harmonic oscillator,

$$\hat{H}_0 = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$$

has eigenvalues $E = \hbar\omega(n_x^2 + n_y^2 + 1)$

and eigenvectors $|n_x, n_y\rangle$

The ground state is $|0,0\rangle$ and is nondegenerate, but the first excited state $E_1 = 2\hbar\omega$ is doubly degenerate – $|1,0\rangle, |0,1\rangle$.

The perturbation may be written in terms of raising/lowering operators,

$$\hat{H}_1 = 2b\,\hat{x}\hat{y} = \frac{\hbar b}{m\omega}(a_x + a_x^\dagger)(a_y + a_y^\dagger)$$

the first order correction to the ground state vanishes,

$$E_0^{(1)} = \frac{\hbar b}{m\omega}\langle 0,0|(a_x + a_x^\dagger)(a_y + a_y^\dagger)|0,0\rangle = \frac{\hbar b}{m\omega}\langle 0,0|1,1\rangle = 0$$

For the degenerate first excited state we must diagonalize the $[\hat{H}_1]$ matrix.

$$\langle 0,1|\hat{H}_1|1,0\rangle = \frac{\hbar b}{m\omega}\langle 0,1|(a_x + a_x^\dagger)(a_y + a_y^\dagger)|1,0\rangle = \frac{\hbar b}{m\omega} = \langle 1,0|\hat{H}_1|0,1\rangle$$

$$\langle 0,1|\hat{H}_1|0,1\rangle = 0 = \langle 1,0|\hat{H}_1|1,0\rangle$$

$$[\hat{H}_1] = \frac{\hbar b}{m\omega}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalue $E^{(1)}_{\pm} = \pm\left(\frac{\hbar b}{m\omega}\right)$

So to first order,

$$E = 2\hbar\omega \pm \frac{\hbar b}{m\omega} = \hbar\omega\left(2 \pm \frac{b}{m\omega^2}\right)$$

11.12 a) Considering only the correction due to $\hat{H}_K$:

$$E^{K}_{n,l} = -\frac{1}{2} mc^2 (Z\alpha)^4 \left(\frac{-3}{4n^4} + \frac{1}{n^3(l+\frac{1}{2})} \right)$$

$$E^{K}_{1s} = -\frac{5}{8} mc^2 (Z\alpha)^4$$

$$E^{k}_{2p} = \frac{-7}{384} mc^2 (Z\alpha)^4$$

$$\left(E^{k}_{2s} = \frac{-13}{128} mc^2 (Z\alpha)^4 \right)$$

(b) In a magnetic field, the perturbing Hamiltonian is,

$$\hat{H}' = \frac{-e}{2mc} \vec{\hat{L}} \cdot \vec{B}$$

the 1s state has $\langle \vec{L} \rangle = 0$ and is unaffected. the splitting of the degenerate 2p states is found by diagonalizing,

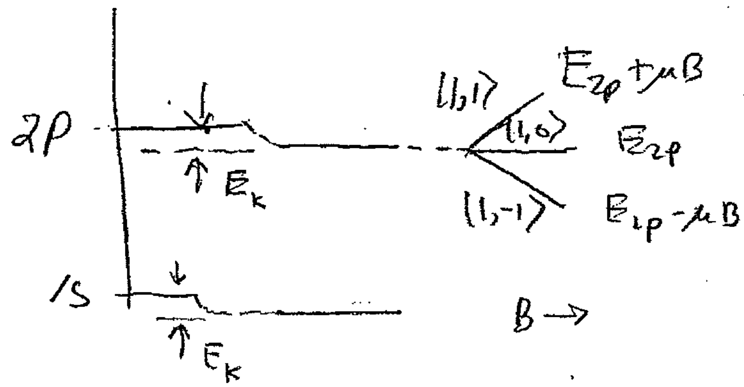
$$\hat{H}' = \frac{eB}{2mc} \hat{L}_z$$

in the Basis $|1,1\rangle, |1,0\rangle, |1,-1\rangle$:

$$[\hat{H}'] = \underbrace{\frac{eB\hbar}{2mc}}_{\mu_B B} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since the matrix is diagonal, we immediately find

$$E_{2p\pm 1} = \pm \mu_B B$$



The resulting 2p→1S transition is split into 3 lines.

11.15 for $\mu_B B \gg mc^2(Z\alpha)^4$ $(B \gg 10^4 \text{ G})$ we can ignore to first order the fine structure corrections. Then $2p$ is 6-fold degenerate, with basis states

$$|1, m_\ell\rangle_z \, |\tfrac{1}{2}, m_s\rangle_z \quad m_\ell = 0, \pm 1; \; m_s = \pm\tfrac{1}{2}$$

However, $\hat{H}'$ is diagonal –

$$\hat{H}' = \frac{\mu_B B}{\hbar}\left(\hat{L}_z + 2\hat{S}_z\right)$$

$$\hat{H}' \, |1, m_\ell\rangle|\tfrac{1}{2}, m_s\rangle = \mu_B B\,(m_\ell + 2m_s)\,|1, m_\ell\rangle|\tfrac{1}{2}, m_s\rangle$$

m_ℓ	m_s	$E^{(1)}/\mu_B B$
1	½	2
1	−½	0
0	½	1
0	−½	−1
−1	½	0
−1	−½	−2