Intermediate Quantum 492: HW #6 Extra

#1) In this problem you will show that in general, the relativistic corrections depend only on the quantum numbers n, j where $|\ell - \frac{1}{2}| \leq j \leq \ell + \frac{1}{2}$
(a) First, show that for the case $\ell > 0$ the total fine structure correction is

$$E^{fs}_{n,j} = a\left[3 - \frac{4n}{j + \frac{1}{2}}\right]$$

where $a \equiv mc^2\alpha^4/8n^4$.
(b) Next show that for $\ell = 0, j = \frac{1}{2}$ the total fine structure is

$$E^{fs}_{n,\frac{1}{2}} = a\left[3 - 4n\right]$$

which is the same as the $\ell > 0, j = \frac{1}{2}$. Therefore, the states $nS_{\frac{1}{2}}$ and $nP_{\frac{1}{2}}$ are degenerate.
#2) Work out the Hamiltonian matrix for the complete $n = 2$ Zeeman effect. Take the external magnetic field B^{ext} to be in the z-direction. The full perturbation Hamiltonian for $n = 2$ is $H^Z = H^{SO} + H^B$ where

$$H^{SO} = a\left[3 - \frac{8}{j + \frac{1}{2}}\right]$$

And with $b \equiv e\hbar B^{ext}/2mc$

$$H^B = b\left(m_\ell + 2m_s\right)$$

In the notation L_{j,m_j} choose the basis order $|i\rangle, i = 1, 8$

$S_{\frac{1}{2},\frac{1}{2}}$, $S_{\frac{1}{2},\frac{-1}{2}}$, $P_{\frac{3}{2},\frac{3}{2}}$, $P_{\frac{3}{2},\frac{-3}{2}}$, $P_{\frac{3}{2},\frac{1}{2}}$, $P_{\frac{1}{2},\frac{1}{2}}$, $P_{\frac{3}{2},\frac{-1}{2}}$, $P_{\frac{1}{2},\frac{-1}{2}}$

So for example $P_{\frac{3}{2},\frac{3}{2}} = |3\rangle = |3/2, 3/2\rangle = |1, 1\rangle |1/2, 1/2\rangle$
Write the matrix $\langle i| \left(H^{SO} + H^B\right) |j\rangle$. It helps to remember that the matrix is Hermitian and real.
You will need Clebsch-Gordon coefficients which you can look up here in the table or find in Griffiths. Two pairs of states are mixed by the external magnetic field. Which pairs are they?