Physics 492
M. Gold

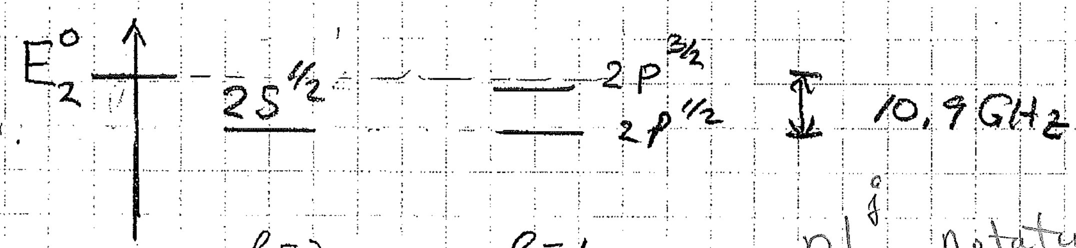
Spring 2017

Lecture 11: H atom fine structure

Relativistic correction $\mathcal{O}(\alpha^4)$

$\vec{J} = \vec{L} + \vec{S}$ spin-orbit coupling $\vec{L}\cdot\vec{S}$

$[\hat{H}, \hat{J}^2] = 0$ $E_{n,j}$ $|l-s| \le j \le l+s$, $\underline{\underline{S=\frac{1}{2}}}$



nL^{j} notation
with $S=\frac{1}{2}$ assumed

$\Delta E\,(2S-1S) = \underline{10.2}\ \text{eV}$

$\Delta E\,(10\,\text{GHz}) = 10^{10}(2\pi)\,6.58\times10^{-16}\ \text{eV}\cdot\text{s} = \underline{4.1\times10^{-5}\ \text{eV}}$

$\alpha^{-2} \approx 20{,}000$ (18769)

Exact solution from Dirac equation:
Gordon, 1928

$$E_{n,j} = mc^2\left[1 + \left(\frac{Z\alpha}{n-(j+\frac{1}{2}) + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}}\right)^2\right]^{-1/2}$$

Expect relativistic correction to be small

$\langle K\rangle = \langle H\rangle - \langle V\rangle$; $\langle V\rangle = -e^2\langle\frac{1}{r}\rangle = \frac{-e^2}{a_0 n^2} = 2E_n$

$\langle K\rangle = \frac{1}{2}mc^2\alpha^2\frac{1}{n^2} \ll mc^2$

$\langle K\rangle_0 = \frac{1}{2}m(c\alpha)^2 = \frac{1}{2}m\langle v^2\rangle$, $\boxed{\sqrt{\langle v^2\rangle} \approx c\alpha}$

A few words on Dirac equation:

Dirac sought relativistic equation (1928)
linear in time:* $E = \pm\sqrt{(mc^2)^2 + (pc)^2}$

$$\hat{H} = \vec{\alpha}\cdot\vec{p}\,c + \beta mc^2 + V$$

α_i, β must be 4×4 matrices (

Dirac "4-spinor" wave function (T = transpose)

$$\psi_D = (\psi_1, \psi_2, \psi_3, \psi_4)^T$$

spin $\frac{1}{2}$ e^-, e^+ $e^+e^- \to \gamma\gamma$ "$E = mc^2$"

spin, antiparticle follow from adding special relativity to Q.M.

* Dirac tried to preserve wave function as conserved probability density. Instead, fermion number is conserved, but necessarily have particle, antiparticle creation, annihilation.

Expand exact solution.

$$E_{nj} \cong mc^2 \left\{ 1 - \frac{1}{2}\frac{(Z\alpha)^2}{n^2}\left[1 + \frac{(Z\alpha)^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)\right]\right\}$$

$$\hat{H} \cong mc^2 + \frac{p^2}{2m} \underbrace{- \frac{p^4}{8m^3c^2}}_{\hat{H}_k} + \underbrace{\frac{e}{2mc}\vec{S}\cdot\vec{B}'}_{\hat{H}_{s.o.}} + \underbrace{\frac{\hbar^2}{8mc^2}\nabla^2 V}_{\hat{H}_D}$$

kinetic spin-orbit Darwin

note $\vec{B}'$ is internal magnetic field due to proton current in electron rest frame.

In external magnetic field, additional piece of Hamiltonian

$$\hat{H}_{ext} = -\vec{\mu}\cdot\vec{B} = \frac{e}{2mc}(\vec{L} + \underset{\uparrow g=2}{2}\vec{S})\cdot\vec{B}$$

Kinetic energy correction

$$K = \sqrt{(pc)^2 + (mc^2)^2} - mc^2$$

$$\cong mc^2\left[1 + \frac{1}{2}\left(\frac{pc}{mc^2}\right)^2 - \frac{1}{8}\left(\frac{pc}{mc^2}\right)^4\right] - mc^2$$

$$= \frac{1}{2}\frac{p^2}{m} - \frac{1}{8}\frac{p^4}{m^3c^2}$$

$[\hat{H}_K, \hat{L}^2] = 0$, $[\hat{H}_K, \hat{L}_z] = 0$ so does not mix states of different l, m:

$$E^{(1)}_{n,l} = -\langle n,l,m | \frac{1}{8}\frac{\hat{p}^4}{m^3c^2} | n,l,m\rangle$$

trick to evaluate $\hat{p}^4$:

$$\left(\frac{\hat{p}^2}{2m}\right)^2 = \left(H_0 + \frac{e^2}{r}\right)^2$$

$$E^{(1)}_{n,l} = -\frac{1}{2mc^2}\langle n l m | \left(\frac{\hat{p}^2}{2m}\right)^2 | n l m\rangle$$

$$= -\frac{1}{2mc^2}\langle n l m | \left(E_n^0 + \frac{e^2}{r}\right)\left(E_n^0 + \frac{e^2}{r}\right) | n l m\rangle$$

$$= -\frac{1}{2mc^2}\left[(E_n^0)^2 + 2E_n^0 \hbar c\alpha \langle \tfrac{1}{r}\rangle + (\alpha\hbar c)^2\langle \tfrac{1}{r^2}\rangle\right]$$

$$\langle \tfrac{1}{r}\rangle = \frac{1}{n^2 a_0} = \frac{\alpha mc}{\hbar^2 n^2}$$

$$\langle \tfrac{1}{r^2}\rangle = \frac{1}{a_0^2}\frac{1}{n^3(l+\frac{1}{2})} = \frac{(\alpha mc)^2}{n^3\hbar^3(l+\frac{1}{2})}$$

giving

$$E^{K(1)}_{n,l} = -\frac{\alpha^4 mc^2}{2}\left[\frac{-3}{4n^4} + \frac{1}{n^3}\frac{1}{(l+\frac{1}{2})}\right]$$

note dependence on l

$$\frac{E^K_{n,l}}{E_n} \propto \alpha^2$$

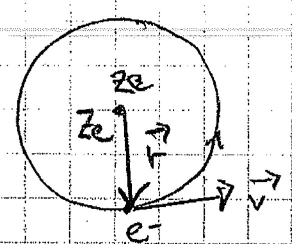
$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta E}{E} \propto \alpha^2 \simeq 10^{-4}$$

Spin Orbit:

Electron orbital motion $|l,m\rangle$ generates magnetic field in electron rest frame.

Classically:

nuclear rest frame



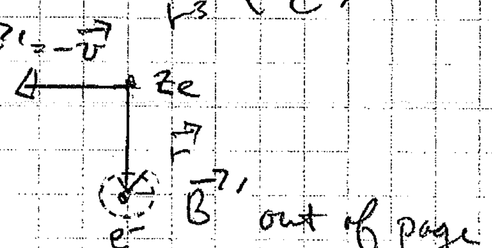
$$\vec{L} = m\vec{r}\times\vec{v} \quad \text{out of page}$$

$$\vec{E} = \frac{Ze\vec{r}}{r^3}$$

In electron's rest frame:

$$\vec{B}' = \left(-\frac{\vec{v}}{c}\right)\times\vec{E} = -\frac{Ze}{r^3}\left(\frac{\vec{v}}{c}\right)\times\vec{r} = \frac{Ze}{mcr^3}\vec{L}$$

e^- rest frame



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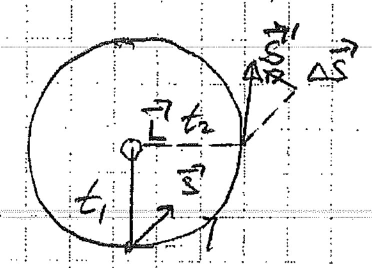
Electron spin experiences a torque in the electron rest frame

$$\left(\frac{d\vec{S}}{dt}\right)_{e^-\text{rest}} = \vec{\mu} \times \vec{B}' = \left(\frac{-ge}{2mc}\right)\vec{S} \times \left(\frac{Ze}{mcr^3}\right)\vec{L}$$

$$= \frac{-Ze^2 g}{2m^2c^2r^3}\left(\vec{S} \times \vec{L}\right)$$

where g is the electron gyromagnetic ratio: $g = 2$ from Dirac equation.

projection of $\vec{S}$ onto orbital plane will precess counter clockwise.



The energy due to this interaction is

$$\hat{H} = -\hat{\vec{\mu}} \cdot \hat{\vec{B}}' = \frac{ge}{2mc}\hat{\vec{S}} \cdot \hat{\vec{B}}'$$

using $g = 2$, this is 2 times bigger than $\hat{H}_{SO}$ from the Dirac equation! The error, first pointed out by Thomas, is a special relativistic effect due to the transformation of the electron rotating frame.

Thomas precession ($\vec{\omega}_T$) (1927)

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{nucleus}} = \left(\frac{d\vec{S}}{dt}\right)_{e^-\ \text{rest}} + \underbrace{\vec{\omega}_T \times \vec{S}}$$

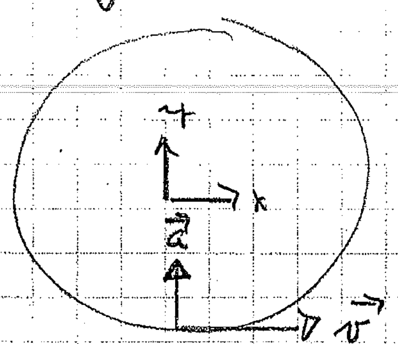
special relativistic effect of transformation

Refer to Taylor and Wheeler, Spacetime physics.

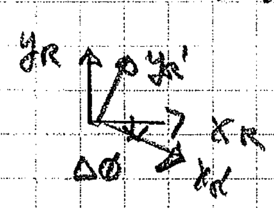
Essentially, the product of "boosts" in different direction is equal to a boost x rotation (general Lorentz transformation)

Consider rotating frame with acceleration $\vec{a}$.

x-y denotes fixed inertial frame



rotating frame will precess clockwise by an amount $\Delta\phi$ per rotation



$$\Delta\phi = \pi\left(\frac{v}{c}\right)^2$$

$$|\vec{\omega}_T| = \Delta\phi\left(\frac{v}{2\pi r}\right) = \frac{v^2}{c^2}\left(\frac{v}{2r}\right) = \frac{1}{2}\frac{v}{c^2}\,a$$

, $a = \frac{v^2}{r}$ for uniform circular motion

$$\vec{\omega}_T = -\frac{1}{2c^2}\,\vec{v}\times\vec{a}$$

Lorentz group has 6 generators: 3 boosts, 3 rotations

Acceleration of e^- in atom is due to nuclear field.

$$\vec{a} = \frac{-e}{m}\vec{\mathcal{E}} = -\frac{Ze^2}{mr^3}\vec{r}$$

$$\vec{\omega}_T = \left(\frac{-1}{2c^2}\right)\vec{v}\times\left(-\frac{Ze^2\vec{r}}{mr^3}\right) = \frac{-e}{2mc}\vec{B}'$$

Correct formula for torque is

$$\left(\frac{d\vec{S}}{dt}\right)_{\text{nuclear}} = \frac{-Ze^2}{2m^2c^2r^3}\left(g\vec{S}\times\vec{L} + \underbrace{\vec{L}\times\vec{S}}_{\text{Thomas term}}\right)$$

$$= \frac{-Ze^2}{2m^2c^2r^3}(g-1)\,\vec{S}\times\vec{L}$$

Since $g=2$, we find $\frac{1}{2}$ of our previous result:

$$\boxed{\hat{H}_{SO} = \frac{Ze^2}{2m^2c^2r^3}(\vec{L}\cdot\vec{S}) = \frac{e}{2mc}\vec{S}\cdot\vec{B}'}$$

In agreement with Dirac.

The constants can be arranged as

$$\boxed{\hat{H}_{SO} = \frac{1}{4}(Z\alpha)^4 mc^2 \left(\frac{a_0}{Zr}\right)^3 \frac{2\,\vec{L}\cdot\vec{S}}{\hbar^2}}$$

showing $O(\alpha^4)$ dependence

$\vec{L}\cdot\vec{S}$ coupling

Hamiltonian will be diagonal in state of total $\hat{J}^2, \hat{J}_z$ where $\vec{J} = \vec{L}+\vec{S}$

$$|\ell, m_\ell\rangle \otimes |\tfrac{1}{2}, m_s\rangle \underset{\text{mix}}{\longleftrightarrow} |j, m_j\rangle$$

$$(2\ell+1)(2\tfrac{1}{2}+1) = 2(2\ell+1) \text{ states}$$

Formally, this is just as with 2-spin system.

$$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{\vec{L}}\cdot\hat{\vec{S}}$$

components of $\vec{L}, \vec{S}$ commute: $[\hat{L}_i, \hat{S}_j] = 0$

Since $\vec{L}\cdot\vec{S}$ is rotationally invariant, we expect

$$[J^2, \vec{L}\cdot\vec{S}] = 0$$

$$[J_z, \vec{L}\cdot\vec{S}] = 0$$

so eigenvalues j, m_j are good quantum number

also $[\hat{L}^2, \hat{\vec{L}}\cdot\vec{S}] = 0$

$$[\hat{S}^2, \hat{\vec{L}}\cdot\vec{S}] = 0$$

However,

$$\left[\hat{L}_z, \vec{L}\cdot\vec{S}\right] = \left[\hat{L}_z, \hat{L}_x\right]\hat{S}_x + \left[\hat{L}_z, \hat{L}_y\right]\hat{S}_y$$

$$= i\hbar\left(\hat{L}_y \hat{S}_x - L_x \hat{S}_y\right) \neq 0$$

similarly,

$$\left[\hat{S}_z, \vec{L}\cdot\vec{S}\right] = i\hbar\left(\hat{L}_x \hat{S}_y - \hat{L}_y \hat{S}_x\right) \neq 0$$

So good quantum number are

$$j,\ m_j,\ l,\ s \quad \text{where } s = \tfrac{1}{2}$$

$\hat{H}_{SO}$ mixes states of same l, s to get $\underline{j = l \pm \frac{1}{2}}$ such that $\underline{m_j = m_l \pm \frac{1}{2}}$

$$|j, m_l + \tfrac{1}{2}\rangle = C_+ |l, m_l\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle + C_- |l, m_l + 1\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|j, m_l - \tfrac{1}{2}\rangle = C_+' |l, m_l - 1\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle + C_-' |l, m_l\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$m_j = m_l \pm \tfrac{1}{2}$$

Constants C are called Clebsch-Gordon (C-G) coefficient and are worked out using SU(2) group theory and looked up in a Clebsch-Gordon table

posted on web page

<u>Example</u> 2p state of Hydrogen

for $n=2$, $l=0,1$ $\hat{H}_{SO}$ will be diagonal in states $|j, m_j\rangle$.

Irrep (irreducible representation) decomposition:

$$\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2} \qquad \text{"} \underset{\sim}{1} \otimes \underset{\sim}{\tfrac{1}{2}} = \underset{\sim}{\tfrac{3}{2}} + \underset{\sim}{\tfrac{1}{2}} \text{"}$$

multiplicities eigenvalues

following the rule $j = 1 \pm \frac{1}{2}$

$$= \begin{cases} 3/2 \\ \frac{1}{2} \end{cases}$$

These <u>multiplets</u> are labeled as $2p^{3/2}$, $2p^{1/2}$
4-plet, 2-plet

indicating n, l, j ($s = \frac{1}{2}$ is assumed)

m_j follow the rule $m_j = -j, -j+1, \ldots, j$

$\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}$ energies can be easily found from

$$\frac{2\vec{L}\cdot\vec{S}}{\hbar^2} = \frac{1}{\hbar^2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right)$$

$$\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}\left|2p^j\right\rangle = \left[j(j+1) - 1(1+1) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right]\left|2p^j\right\rangle$$

$$= \left[j(j+1) - \tfrac{11}{4}\right]\left|2p^j\right\rangle \equiv \lambda_j \left|2p^j\right\rangle$$

$$\boxed{\begin{aligned} \text{we find} \quad \lambda_{3/2} &= 1 \\ \lambda_{\frac{1}{2}} &= -2 \end{aligned}}$$

To find eigenstates, we can look up C-G coefficients in table or explicitly diagonalise $\vec{L}\cdot\vec{S}$.

To see how this goes,

$$|\tfrac{3}{2}, \pm\tfrac{3}{2}\rangle = |1, \pm 1\rangle\,|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$

we must diagonalize two, 2x2 submatrices

for $\quad |1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle\,,\ |1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \quad m_j = +\tfrac{1}{2}$

and for $\quad |1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle,\ |1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \quad m_j = -\tfrac{1}{2}$

$\boxed{m_j = +\tfrac{1}{2} \text{ case:}}$ use basis $|1\rangle = |1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$

$|2\rangle = |1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$

write $$\frac{2\vec{L}\cdot\vec{S}}{\hbar^2} = \frac{1}{\hbar^2}\left[\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + 2\hat{L}_z\hat{S}_z\right]$$

$$\langle 1|\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}|1\rangle = \langle 1|\frac{2}{\hbar^2}L_z S_z|1\rangle = 0$$

$$\langle 2|\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}|2\rangle = \langle 2|\frac{2}{\hbar^2}\hat{L}_z\hat{S}_z|2\rangle = +1$$

$$\langle 2|\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}|1\rangle = \langle 2|\frac{1}{\hbar^2}\hat{L}_+\hat{S}_-|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= \sqrt{2}\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)} = \sqrt{2}$$

giving $\left[\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}\right]_{1,2} = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$

eigenvalue equation

$$\begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \lambda \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\det \begin{vmatrix} -\lambda & \sqrt{2} \\ \sqrt{2} & -1-\lambda \end{vmatrix} = \lambda(1+\lambda) - 2 = 0$$

$$(\lambda - 1)(\lambda + 2) = 0 \qquad \lambda = 1, -2$$

and eventually eigenvectors

$$\lambda = 1 \qquad \left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\,|1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

$$\lambda = -2 \qquad \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{1}{3}}\,|1,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\,|1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

from CG table: (with Condon-Shortley phase convention)

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\,|1,-1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = +\sqrt{\tfrac{1}{3}}\,|1,0\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\,|1,-1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

By rotational invariance, all states with same j have same energy $[\hat{H}_{SO}, \hat{J}_z] = 0$

$$j = 3/2 \longrightarrow \lambda = 1$$

$$j = \tfrac{1}{2} \longrightarrow \lambda = -2$$

as on p.12

<u>note</u> in each term $m_j = m_\ell + m_s$

so in diagonal basis

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle\ |\tfrac{3}{2},\tfrac{1}{2}\rangle\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle\ |\tfrac{3}{2},-\tfrac{3}{2}\rangle\ |\tfrac{1}{2},\tfrac{1}{2}\rangle\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\left[\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}\right] = \text{diag}(1,1,1,1,-2,-2)$$

<u>Spin-orbit energy</u>

$$\hat{H}_{SO} = \frac{1}{4}mc^2(Z\alpha)^4\left(\frac{a_0}{Zr}\right)^3\left(\frac{2\vec{L}\cdot\vec{S}}{\hbar^2}\right)$$

$$\left\langle\left(\frac{a_0}{Zr}\right)^3\right\rangle_{n,\ell} = \left.\frac{1}{n^3\ell(\ell+\frac{1}{2})(\ell+1)}\right|_{\substack{n=2\\ \ell=1}} = \frac{1}{2^3\cdot 3} = \frac{1}{24}$$

$$E_{SO}^{(1)} = \left(\frac{1}{96}\right)mc^2(Z\alpha)^4\begin{cases}1 & j=3/2\\ -2 & j=1/2\end{cases}$$

Next, Darwin term.

<u>Darwin term</u> (Charles Darwin's Grandson)

$$\hat{H}_D = \frac{\hbar^2}{8m^2c^2} e\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)$$

where $\vec{\mathcal{E}} = -\vec{\nabla}\phi$, $V = -e\Phi$

$$e\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right) = \left(e\vec{\nabla}\right)\left(-\vec{\nabla}\phi\right) = \nabla^2 V$$

more to the point, $\vec{\nabla}\cdot\vec{\mathcal{E}} = 4\pi\rho = Ze\,4\pi\,\delta^3(\vec{r})$

$$\hat{H}_D = \frac{\hbar^2\pi}{2m^2c^2}\,\frac{Ze^2}{a_0^3}\left(\frac{a_0^3}{Z^3}\delta^3(\vec{r})\right)$$

with $e^2 = \hbar c\alpha$, $a_0 = \frac{\hbar}{mc\alpha}$

$$\hat{H}_D = \frac{\pi}{2}\,mc^2(Z\alpha)^4\left(\frac{a_0}{Z}\right)^3\delta^3(\vec{r})$$

only $l=0$ state are non-zero origin

$$E^{(1)}_{D\,n,0} = \langle\hat{H}_D\rangle_{n,0} = \frac{\pi}{2}(mc^2)(Z\alpha)^4\left(\frac{a_0}{Z}\right)^3\left|\psi_{n,0}(0)\right|^2$$

$$\left|\psi_{n,0}(0)\right|^2 = \left(\frac{Z}{na_0}\right)^3\frac{1}{\pi}$$

giving

$$E^{(1)}_{D\,n,0} = \left(\frac{1}{2n^3}\right) mc^2 (Z\alpha)^4 \Big|_{n=2} = +\frac{1}{16} mc^2 (Z\alpha)^4$$

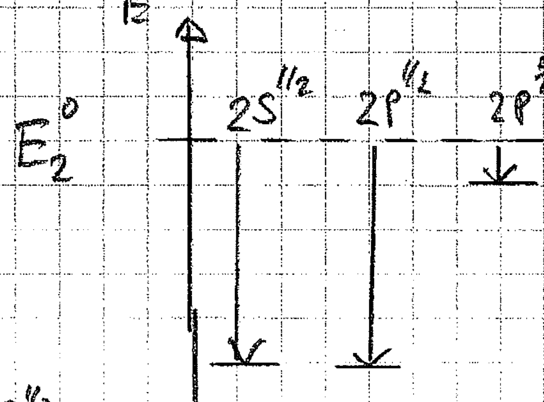
N=2 fine Structure

$$E^{(1)}_{2S^{1/2}} = E_K + E_D = \underbrace{\left(\frac{-13}{128} + \frac{1}{16}\right)}_{-5/128} mc^2 (Z\alpha)^4$$

$$E^{(1)}_{2P^{3/2}} = E_K + E_{SO} = \underbrace{\left(\frac{-7}{384} + \frac{1}{96}\right)}_{-1/128} mc^2 (Z\alpha)^4$$

$$E^{(1)}_{2P^{1/2}} = E_K + E_{SO} = \underbrace{\left(\frac{-7}{384} - \frac{2}{96}\right)}_{-5/128} mc^2 (Z\alpha)^4$$

as for exact solution, E depends only on j !



$$\Delta E_{fine} = \frac{4}{128} mc^2 (Z\alpha)^4$$

$$\Delta\nu_f = \frac{\Delta E}{h} = 10.9 \text{ GHz}$$

Exp. $2S^{1/2}$, $2P^{1/2}$ split by Lamb shift $-\ \Delta\nu_L = 1.057$ GHz