

Lecture 13: Zeeman Effect

refer to Griffiths p. 244

H-atom in external magnetic field ( $\vec{B} = B \hat{e}_z$ )

$$\hat{H}' = -\vec{\mu} \cdot \vec{B}$$

$$\vec{\mu} = -\left(\frac{e\hbar}{2mc}\right) \left(\frac{\vec{L} + g\vec{S}}{\hbar}\right)$$

$g = 2$  point-particle gyromagnetic ratio

$$\mu_B = \frac{e\hbar}{2mc} = 5.79 \times 10^{-9} \text{ eV/G} \quad \text{Bohr magneton}$$

$$= 5.79 \times 10^{-5} \text{ eV/T}$$

$$\mu_B \propto \text{charge/mass}$$

With  $\vec{B} = B \hat{e}_z$   $\hat{H}' = \mu_B B \left(\frac{\hat{L}_z + 2\hat{S}_z}{\hbar}\right)$

Compare  $\langle \hat{H}' \rangle$  to  $\langle \hat{H}_{50} \rangle$

$$\langle \hat{H}_{50} \rangle_{l=2} = \frac{1}{96} mc^2 \alpha^4 \begin{cases} 1 & j=3/2 \\ -2 & j=1/2 \end{cases}$$

$\langle \hat{H}' \rangle = \langle \hat{H}_{50} \rangle$  when

$$B \approx \frac{\frac{1}{96} mc^2 \alpha^4}{\mu_B} = \frac{10^{-2} (5 \times 10^5 \text{ eV}) \left(\frac{1}{137}\right)^4}{6 \times 10^{-9} \text{ eV/G}}$$

$$= 5 \times 10^3 \text{ G} = 0.5 \text{ T}$$

Then we have 3 regimes:

$B \ll T$  weak

$B \gg T$  strong

$B \sim T$  intermediate

Weak  $\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{r} + \hat{H}_{so}$

unperturbed states

$|\phi^0\rangle = |n, l, \frac{1}{2}, j, m_j\rangle$

$\langle \hat{H}' \rangle = \mu_B B \left[ \frac{\langle \hat{J}_z \rangle}{\hbar} + \frac{\langle \hat{S}_z \rangle}{\hbar} \right]$

when we used  $\vec{L} + 2\vec{S} = \vec{J} + \vec{S}$

Since  $|\phi^0\rangle$  are not eigenstates of  $\hat{S}_z$  we need Clebsch-Gordan coefficients

$|\underbrace{l \pm \frac{1}{2}}_j, l, \frac{1}{2}, m_j\rangle =$

$\left( \frac{l \pm m_j + \frac{1}{2}}{2l+1} \right)^{1/2} |l, m_j - \frac{1}{2}\rangle | \frac{1}{2}, \frac{1}{2} \rangle$

$+ \left( \frac{l \mp m_j + \frac{1}{2}}{2l+1} \right)^{1/2} |l, m_j + \frac{1}{2}\rangle | \frac{1}{2}, -\frac{1}{2} \rangle$

then

$$\langle \hat{S}_z \rangle = \frac{1}{2} \left[ \frac{l \pm m_j + \frac{1}{2}}{2l+1} - \frac{l \mp m_j + \frac{1}{2}}{2l+1} \right]$$

$$= \frac{\pm m_j}{2l+1} \quad \text{where } \pm \text{ corresponds to } j = l \pm \frac{1}{2}$$

adding this to  $\langle \hat{J}_z \rangle$  term,

$$\langle \hat{H}' \rangle_{j, l, m_j} = \mu_B B m_j \left( 1 \pm \frac{1}{2l+1} \right)$$

degeneracy removed

Landé g-factor\*

Landé g-factor can also be written  
(see Griffiths)

$$g_L(j, l) = \frac{1 + j(j+1) - l(l+1) + \frac{3}{4}}{2j(j+1)}$$

for  $l=0$  there is no  $\pm$  since  $j = \frac{1}{2}$  only  
 $l-5/4 \leq l+s$ . In this case  $g_L = 2$

Example  $n=2$

$$S^{1/2} \quad \langle H' \rangle = 2 \mu_B B m_j = \pm \mu_B$$

$$P^{1/2} \quad (j = 1 - \frac{1}{2})$$

$$\begin{aligned} \langle H' \rangle &= \mu_B B m_j \left(1 - \frac{1}{3}\right) \\ &= \pm \frac{1}{3} \mu_B B \end{aligned}$$

$$P^{3/2} \quad (j = 1 + \frac{1}{2})$$

$$\langle H' \rangle = \mu_B B m_j \left(1 + \frac{1}{3}\right) = \frac{4}{3} \mu_B B m_j$$

$$m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$$

Note - energy splitting  $\propto B$ .

### Strong field

Consider  $\hat{H}_{so}$  as perturbation to

$$\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{r} + \mu_B B \left( \frac{\hat{L}_z + 2\hat{S}_z}{\hbar} \right)$$

$$|\phi_0\rangle = |E, l, m_l\rangle |1/2, m_s\rangle$$

$$E_0 = -\frac{1}{2n^2} mc^2 \alpha^2 + \mu_B B (m_l + 2m_s)$$

We don't use  $\hat{J}^2, \hat{J}_z$  basis because these are not conserved due to the external torque.

Since there is no longer any degeneracy, we can evaluate  $\langle \vec{L} \cdot \vec{S} \rangle$  using non-degenerate perturbation theory. Write

$$\vec{L} \cdot \vec{S} = \frac{1}{2} (\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+) + \hat{L}_z \hat{S}_z$$

$$\langle \vec{L} \cdot \vec{S} \rangle = 0 + \langle L_z S_z \rangle = \hbar^2 m_l m_s$$

total first order correction is then

$$E_{fs}^1 = \left( \frac{1}{2n^3} \right) mc^2 \alpha^4 \left\{ \frac{3}{4n} - X \right\}$$

where for  $l=0, X=1$

$$l > 0, X = \frac{l(l+1) - m_l m_s}{l(l + \frac{1}{2})(l+1)}$$

See next page

$$E = -\frac{1}{2n^2} m c^2 \alpha^2 + \mu_B B (m_l + 2m_s) + E'_{fs}$$

dependence on B:  $\frac{1}{\mu_B} \frac{dE}{dB} = m_l + 2m_s = m_l \pm 1$

$l$	$m_l$	$m_l \pm 1$	$n=2$
0	0	1 (-1)	splits into 5
1	1	2 (0)	line
	0	1 (-1)	
	-1	0 (-2)	
		$\uparrow$	$m_s = -\frac{1}{2}$
			$m_s = +\frac{1}{2}$

In intermediate case:

diagonalize  $[-\vec{\mu}_B \cdot \vec{B}] + [H_{fs}]$  in unperturbed basis

$$|n, l, m_l\rangle | \frac{1}{2}, m_s \rangle$$

$$\text{degeneracy} = 2n^2$$

for  $n=2$ ,  $8 \times 8$  matrix. label as  $L(j, m_j)$

$$P(\frac{3}{2}, \frac{1}{2}), P(\frac{1}{2}, \frac{1}{2}) \text{ mix}$$

$$P(\frac{3}{2}, -\frac{1}{2}), P(\frac{1}{2}, -\frac{1}{2}) \text{ mix}$$

no spin flip

## Energy in Strong Zeeman

$$E_K^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left( \frac{3}{4n} - l + \frac{1}{2} \right)$$

$$E_{SO}^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left( \frac{m_e m_s}{l(l+1)(l+\frac{1}{2})} \right)$$

$$E_D^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} (-1)$$

$$l=0: E_K^{(1)} + E_D^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left( \frac{3}{4n} - 2 + 1 \right)$$

$$l>0: E_K^{(1)} + E_{SO}^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left( \frac{3}{4n} - \frac{l(l+1) - m_l m_s}{l(l+1)(l+\frac{1}{2})} \right)$$

$$E = -\frac{1}{2} \frac{mc^2 (Z\alpha)^2}{n^2} + \mu_B B (m_l + m_s) + E_{fs}^{(1)}$$

$$E_{fs}^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left( \frac{3}{4n} - x \right)$$

$$l=0 \quad x=1$$

$$l>0 \quad x = \frac{l(l+1) - m_l m_s}{l(l+1)(l+\frac{1}{2})}$$

Zeeman recap degenerate perturbation theory

$$\hat{H}_{Hy} = \frac{\hbar^2}{2\pi} - \frac{e^2}{r} \quad |n, l, m_l, s, m_s\rangle \quad 2n^2 \text{ degenerate}$$

$$\hat{H}_{fs} = \hat{H}_k + \hat{H}_D + \hat{H}_{so} \quad |n, j, m_j, l, s\rangle$$

multiplets  $2j+1$  degenerate

$$\hat{H}_z = \mu_B B \left( \frac{\hat{L}_z + 2\hat{S}_z}{\hbar} \right) \quad \text{Zeeman}$$

weak  $B \ll T$   $\hat{H}_z$  is perturbation

$$\langle \hat{H}_z \rangle_{n, j, m_j, l, s} = \mu_B B m_j \left( 1 \pm \frac{1}{2l+1} \right) \text{ diagonal}$$

where for  $l=0$ , only +  $2j+1$  degeneracy removed

strong  $B \gg T$   $\hat{H}_{fs}$  perturbation

$$\langle \hat{H}_{fs} \rangle_{n, l, m_l, s, m_s} = \frac{1}{2n^3} m_e^2 \alpha^4 \left\{ \frac{3}{4n} - x \right\}$$

diagonal

$$x = \begin{cases} 1 & l=0 \\ \frac{l(l+1) - m_l m_s}{l(l+1)(l+\frac{1}{2})} & l>0 \end{cases}$$

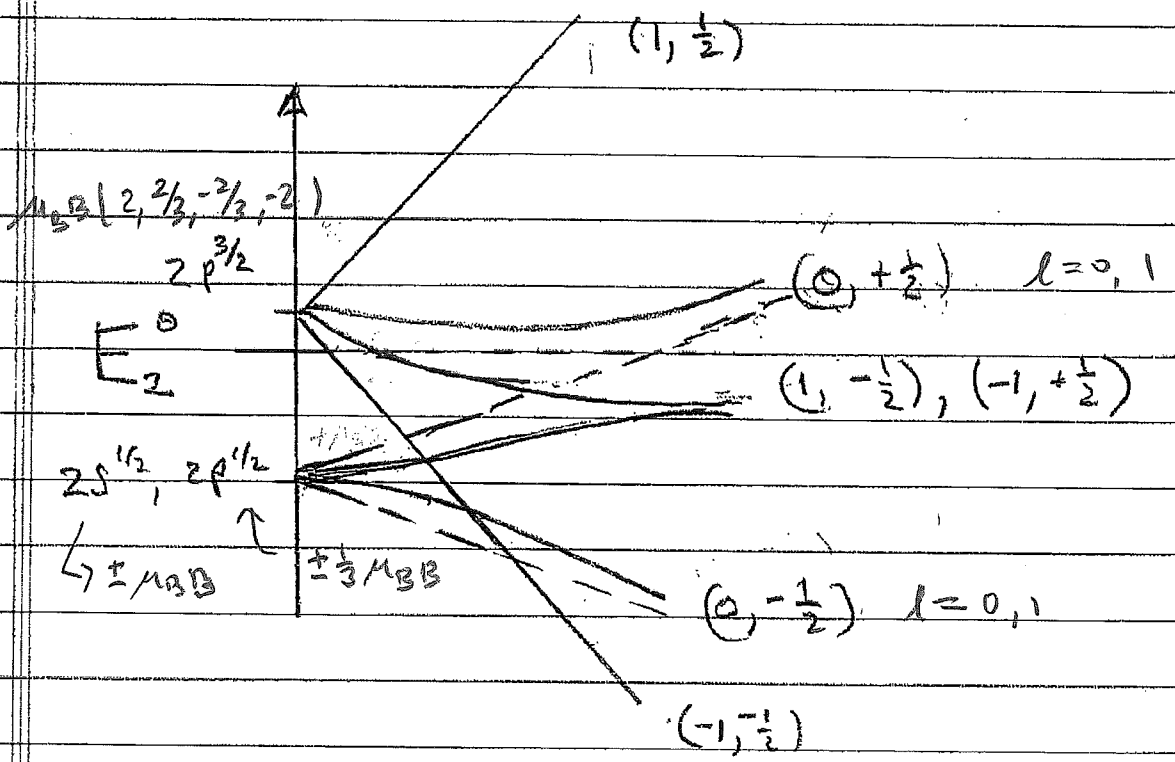
for  $n=2$  degenerate 8  $\rightarrow$  5 lines

intermediate  $B \sim T$  diagonalize  $\langle \hat{H}_z + \hat{H}_{fs} \rangle_{n, l, m_l, s, m_s}$  basis



Sketch for  $n=2$   
 Griffiths fig 6.16 p 249

notation  $(m_l, m_s)$



weak

strong

Strong splittings

$l$	$m_l$	$m_s$	$m_l \pm 2m_s$
0	0	$\frac{1}{2}$	1
		$-\frac{1}{2}$	-1
1	1	$\frac{1}{2}$	2
		$-\frac{1}{2}$	0
	0	$\frac{1}{2}$	1
		$-\frac{1}{2}$	-1
-1	$\frac{1}{2}$	0	
	$-\frac{1}{2}$	-2	

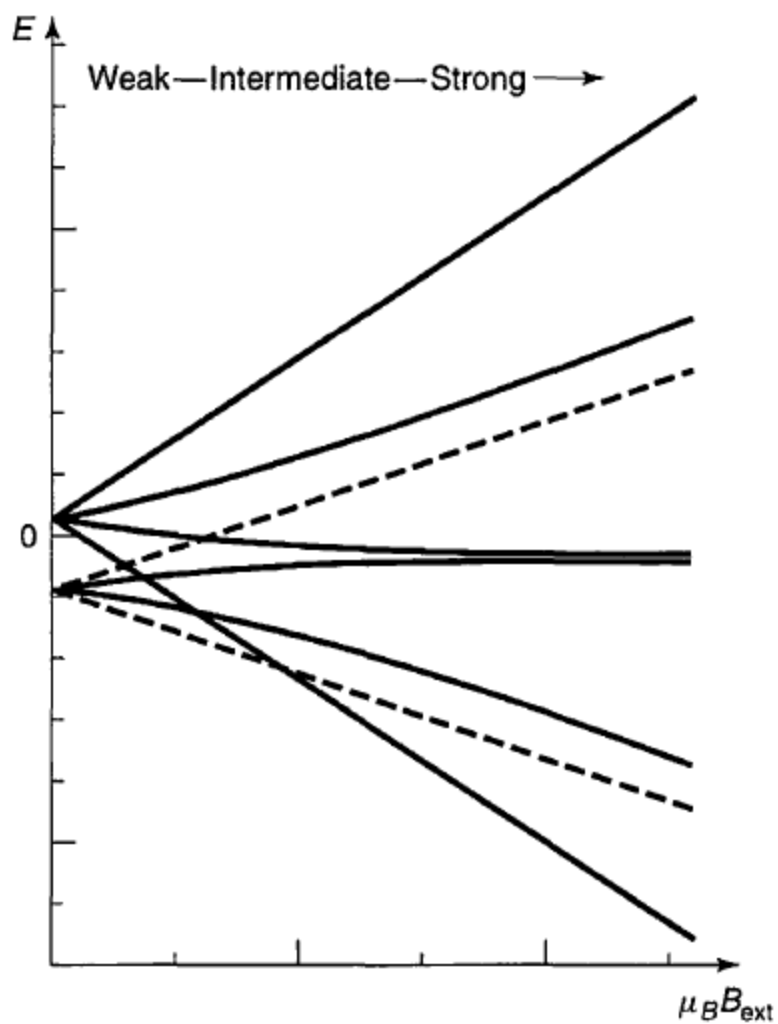


FIGURE 6.12: Zeeman splitting of the  $n = 2$  states of hydrogen, in the weak, intermediate, and strong field regimes.