Physics 492
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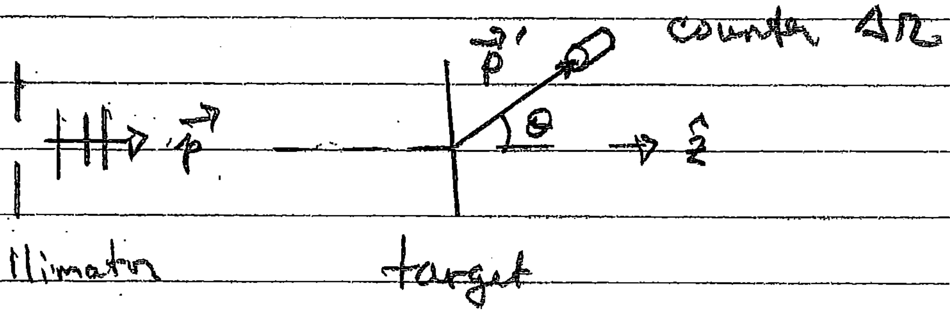
Spring 2010

Lec 17: Scattering Theory I



Beam of collimated particles is approximate plane wave ($p=\hbar k$) $\phi_{\vec k}(\vec r) = e^{i\vec k\cdot\vec r} = \langle\vec r|k\rangle$

$\langle k'/k\rangle = (2\pi)^3\delta^3(\vec k'-\vec k)$

Quantity of interest is the differential cross section

$$d\sigma = \left(\frac{d\sigma}{d\Omega}\right)d\Omega \equiv \frac{\text{Rate scattered into } d\Omega \text{ at } \theta,\phi}{\text{incident flux}}$$

flux ≡ # particles incident / area / time

flux measured, e.g. $\dfrac{\text{current}}{q\,(\text{collimator area})}$

By dividing out flux, what is left is intrinsic to interaction (scattering potential)

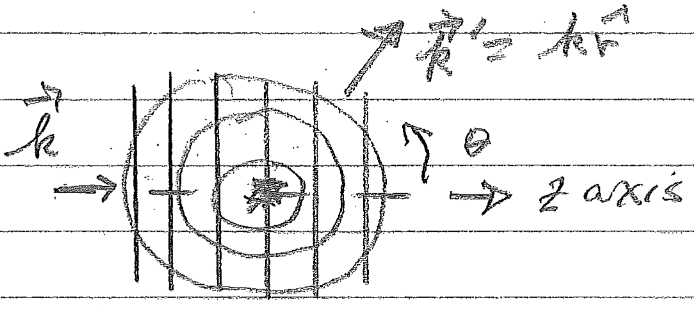
Good reference on non-relativistic scattering is J.R. Taylor Scattering Theory.

σ is dimensionally an area ⊥ to beam. therefore invariant under Lorentz boosts in beam direction

Elastic Scattering

In the cm frame (or for a fixed force center) only the direction of the momentum changes in the collision: $|\vec{k}'| = |\vec{k}|$

Time independent description:



We cannot observe the scattering in the shaded region of size $ƛ = \frac{1}{k}$. Observations are in the asymptotic region $r >> ƛ$

$$\psi \underset{r\to\infty}{\sim} \psi_{in} + \psi_{sc}$$

where $\psi_{in} = Ae^{ikz}$ incident plane wave

$\psi_{sc} \sim A\frac{e^{ikr}}{r} f(\theta,\phi)$ outgoing spherical wave

We will derive this asymptotic form rigorously later. $f(\theta,\phi)$ is called the scattering amplitude.

Note: probability is conserved by destructive interference in the forward ($\theta \approx 0$) direction recall probability flux:

$$\vec{j} = \frac{\hbar}{2\mu i}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right) = \frac{\hbar}{\mu} \mathrm{Im}\left(\psi^* \vec{\nabla}\psi\right)$$

$$\vec{j}_{in} = \frac{\hbar k}{\mu} |A|^2 \hat{z}$$

$$\vec{j}_{sc} \underset{r \to \infty}{\sim} \frac{\hbar k}{\mu r^2} |A|^2 |f|^2 \hat{r}$$

$$\left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{(\vec{j}_{sc} \cdot \hat{r})\, r^2 d\Omega}{|\vec{j}_{in}|} = |f|^2 d\Omega$$

$$\frac{d\sigma}{d\Omega} = |f|^2$$ scattering amplitude squared

Note: since A cancels in the ratio of fluxes we take $A=1$ from now on.

Formal Development from Integral Equation

time independent equation

$$\left(\frac{-\hbar^2}{2\mu}\nabla^2 + V(\vec r)\right)\psi(\vec r) = E\,\psi(\vec r)$$

① $(\nabla^2+k^2)\,\psi = \frac{2\mu}{\hbar^2}V\psi \quad , \; k \equiv \frac{\sqrt{2mE}}{\hbar}$

plane wave solution* $\phi_{\vec k}(\vec r) = e^{i\vec k\cdot\vec r}$ satisfies free particle equation

$$(\nabla^2+k^2)\,\phi_{\vec k}(\vec r) = 0$$

If we knew how to define the inverse operator $(\nabla^2+k^2)^{-1}$ we get a formal solution equivalent to ①

$$\psi = \phi + \frac{2\mu}{\hbar^2}(\nabla^2+k^2)^{-1}V\psi$$

formal solution because the unknown ψ appears on both sides. We could try an iterative solution (perturbative expansion).

Green's function or propagator is defined as

$$\left(\nabla^2_{\vec r} + k^2\right)G(\vec r,\vec r\,') = \delta^3(\vec r-\vec r\,')$$

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* in this section it is convenient to use the plane wave normalization $\langle\vec k'|\vec k\rangle = (2\pi)^3\,\delta^3(\vec k'-\vec k)$

Then the formal solution is

$$\psi(\vec{r}) = \phi_{\vec{k}}(\vec{r}) + \frac{2\mu}{\hbar^2}\int d^3\vec{r}\,'\, G(\vec{r},\vec{r}\,')\, V(\vec{r}\,')\,\psi(\vec{r}\,')$$

The explicit Green's function is found by Fourier transformation.†

$$(\nabla^2 + k^2)\, G(\vec{r},0) = \delta^3(\vec{r})$$

$$\delta^3(\vec{r}\,') = \left(\frac{1}{2\pi}\right)^3 \int d^3\vec{q}\, e^{i\vec{q}\cdot\vec{r}}$$

$$G(\vec{r},0) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3\vec{q}\, e^{i\vec{q}\cdot\vec{r}}\, \tilde{G}(\vec{q})$$

then substitution gives

$$(k^2 - q^2)\, \tilde{G}(\vec{q}) = \left(\frac{1}{2\pi}\right)^{3/2}$$

$$\tilde{G}(\vec{q}) = \left(\frac{1}{2\pi}\right)^{3/2} \left(\frac{1}{k^2 - q^2}\right)$$

Inverse Fourier transform

$$G(\vec{r}) = \left(\frac{1}{2\pi}\right)^3 \int d^3\vec{q}\, e^{i\vec{q}\cdot\vec{r}} \left(\frac{1}{k^2 - q^2}\right)$$

for the integral, $\vec{r}$ is fixed

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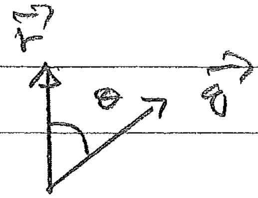
† The free particle propagator from Ch. 8 is the Green's function

$$\left(i\hbar\frac{\partial}{\partial t} - \left(-\frac{\hbar^2}{2m}\nabla_r^2\right)\right) G(\vec{r},t;\vec{r}\,',t') = \delta^3(\vec{r}-\vec{r}\,')\,\delta(t-t')$$



$\vec{q}\cdot\vec{r} = qr\cos\theta$

define $c \equiv \cos\theta$

$$G(\vec{r}) = \left(\frac{1}{2\pi}\right)^3 (2\pi)\int_0^\infty q^2\,dq \int_{-1}^{+1} dc\, e^{iqrc}\left(\frac{1}{k^2-q^2}\right)$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{1}{r}\int_0^\infty q^2\,dq\left(\frac{1}{k^2-q^2}\right)\frac{2}{q}\sin qr$$

$$= \frac{1}{2\pi^2}\,\frac{1}{r}\int_0^\infty \frac{q\,dq\,\sin qr}{k^2-q^2} = \frac{1}{4\pi^2 i}\,\frac{1}{r}\int_0^\infty \frac{q\,dq}{k^2-q^2}\left(e^{iqr}-e^{-iqr}\right)$$

$$= \frac{1}{4\pi^2 i r}\left\{\int_0^\infty \frac{q\,dq}{k^2-q^2}\,e^{iqr} - \int_0^{-\infty}\frac{q\,dq}{k^2-q^2}\,e^{iqr}\right\}$$

$$= \frac{i}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{q\,dq\,e^{iqr}}{q^2-k^2}$$

Integral diverges at $q = \pm k$.

The choice of how to define this integral corresponds to the choice of boundary conditions.

Outgoing spherical wave Green's function is

$$\boxed{G_{out}(\vec{r}) = -\frac{1}{4\pi}\,\frac{e^{ikr}}{r}}$$

Formal Integral equation is

$$\psi(\vec{r}) = \phi_k(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int d^3\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi(\vec{r}\,')$$

<u>Asymptotic limit</u>. $|\vec{r}-\vec{r}\,'| \to r$ in denominator, but in exponential

$$|\vec{r}-\vec{r}\,'| = \left(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'\right)^{1/2} \doteq r - \hat{r}\cdot\vec{r}\,'$$

$$e^{ik|\vec{r}-\vec{r}\,'|} = e^{ikr} e^{-i\vec{k}'\cdot\vec{r}\,'} \text{, where } \vec{k}' = k\hat{r}$$

Then we get the promised asymptotic form

$$\psi(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} \underbrace{\left[\frac{-\mu}{2\pi\hbar^2}\int d^3\vec{r}\,' e^{-i\vec{k}'\cdot\vec{r}\,'} V(\vec{r}\,')\psi(\vec{r}\,')\right]}_{f(\theta,\varphi)}$$

<u>Note</u>: Convergence of the integral requires $V(r) \to 0$ faster than $1/r$. For Coulomb potential, the plane wave approximation fails. In practice, the Coulomb potential is always <u>screened</u> at large r. Mathematically, we can introduce a convergence factor e^{-mr} and take the limit $m \to 0$ after doing the integral

Mathematical Interlude: Green's function and contour integration

$f(z)$ where $z = x + iy$ is analytic if

$f(z) = U(x,y) + iV(x,y)$ satisfies Cauchy-Riemann equations

$$\partial_x U = \partial_y V$$
$$\partial_x V = -\partial_y U$$

$$\oint_C f(z)\,dz = 0 \quad \text{any closed contour } C$$

Cauchy's integral formula

$$\oint_C \frac{f(z)\,dz}{\underset{\substack{\uparrow \\ \text{pole}}}{(z - z_0)}} = 2\pi i \underset{\substack{\uparrow \\ \text{residue}}}{f(z_0)} \qquad C \text{ contains } z_0$$

Method can often be applied to evaluate real integrals.

simple example: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

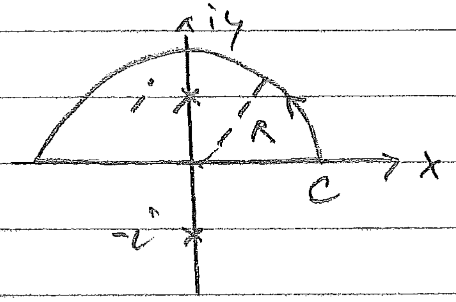
let $x = \tan\theta$, $dx = d\theta/\cos^2\theta$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos^2\theta}\left(\frac{1}{1+\tan^2\theta}\right) = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

By method of contour integration:

$$\frac{1}{1+z^2} = \left(\frac{1}{z+i}\right)\left(\frac{1}{z-i}\right)$$

Complex plane

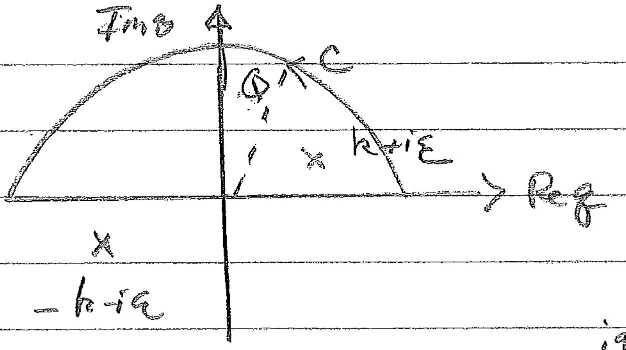


$$\oint \frac{dz}{1+z^2} = 2\pi i\left(\frac{1}{2i}\right) = \pi = \lim_{R\to\infty} \overset{0}{\cancel{\int_0^{\pi} \frac{R\,d\theta}{1+R^2 e^{2i\theta}}}} + \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Evaluation of $G_{out}(\vec{r})$

$$G(\vec{r}) = \frac{i}{4\pi^2 r}\int_{-\infty}^{\infty} \frac{q\,dq\,e^{iqr}}{(q+k)(q-k)}$$

Complex q plane



choice of contour for $G_{out}(\vec{r})$

$$G_{out}(\vec{r}) = \lim_{\substack{\epsilon\to 0\\ Q\to\infty}} \frac{i}{4\pi^2 r}\oint_C \frac{q\,dq\,e^{iqr}}{(q+k+i\epsilon)(q-k-i\epsilon)}$$

$$= \frac{i}{4\pi^2 r}(2\pi i)\,\frac{k e^{ikr}}{2k} = -\frac{1}{4\pi}\,\frac{e^{ikr}}{r}$$