

Lec 18: Scattering II

Approximation methods - Born series, partial waves

Born Series iterative solution to integral equation

$$\psi(\vec{r}) = \phi_k(\vec{r}) + \frac{2\mu}{\hbar^2} \int d^3r' G_+(\vec{r}, \vec{r}') V(\vec{r}') \psi(\vec{r}')$$

$$G_+ = -\frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \quad \text{outgoing spherical wave}$$

$$\psi = \phi + \psi_{sc} \quad \text{iterative solution}$$

$$\psi_{sc}^{(1)} = \frac{2\mu}{\hbar^2} \int d^3r' G_+(\vec{r}, \vec{r}') V(\vec{r}') \phi_k(\vec{r}') \quad (\text{Born})$$

$$\begin{aligned} \psi_{sc}^{(2)} = & \frac{2\mu}{\hbar^2} \int d^3r' G_+(\vec{r}, \vec{r}') V(\vec{r}') \\ & \times \left(\frac{2\mu}{\hbar^2} \right) \int d^3r'' G_+(\vec{r}', \vec{r}'') V(\vec{r}'') \phi_k(\vec{r}'') \end{aligned}$$

2nd order

term $\psi_{sc}^{(n)}$ contains n factors of V .

To extract scattering amplitude, take asymptotic form of G_+ :

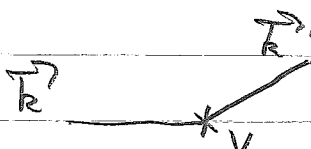
$$G_+(\vec{r}, \vec{r}') \sim \left(-\frac{1}{4\pi}\right) \frac{e^{i\vec{k}\vec{r} - i\vec{k}'\vec{r}'}}{r}$$

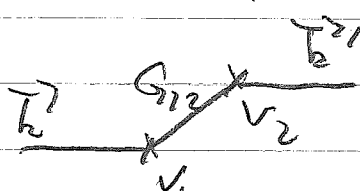
$$\begin{aligned} f^{(1)} &= \frac{2\mu}{\hbar^2} \left(-\frac{1}{4\pi}\right) \int d^3\vec{r}' e^{-i\vec{k}'\vec{r}'} V(\vec{r}') \phi_{\vec{k}}(\vec{r}') \\ &\quad \phi_{\vec{k}}^*(\vec{r}') = \langle \vec{k}' | \vec{r}' \rangle \quad * \\ &= \frac{2\mu}{\hbar^2} \left(-\frac{1}{4\pi}\right) \langle \vec{k}' | V | \vec{k} \rangle \end{aligned}$$

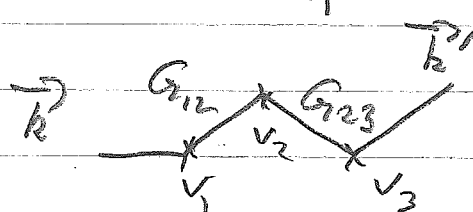
$$f^{(2)} = \frac{2\mu}{\hbar^2} \left(-\frac{1}{4\pi}\right) \int d^3\vec{r}' e^{-i\vec{k}'\vec{r}'} V(\vec{r}') \times$$

$$\left(\frac{2\mu}{\hbar^2}\right) \int d^3\vec{r}'' G_+(\vec{r}', \vec{r}'') V(\vec{r}'') \phi_{\vec{k}}(\vec{r}'')$$

$$= \left(\frac{2\mu}{\hbar^2}\right)^2 \left(-\frac{1}{4\pi}\right) \langle \vec{k}' | V_2 G_{12} V_1 | \vec{k} \rangle$$

pictorially  Born

 2nd order

 3rd order

* Again, normalized as $\langle \vec{k}' | \vec{k} \rangle = (2\pi)^3 \delta^3(\vec{k}' - \vec{k})$

Evaluating the Born Amplitude

Assume a spherically symmetric potential

$$f^{(1)}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int d^3\vec{r}' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} V(r')$$

Fourier transform of potential

Let $\vec{q} \equiv \vec{k} - \vec{k}'$ momentum transfer in collision

$$f^{(1)}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int_0^\infty r'^2 dr' V(r') 2\pi \underbrace{\int_{-1}^{+1} d\cos\theta e^{iqr'}}_{\frac{2}{qr'} \sin(qr')}$$

$$\boxed{f^{(1)}(\theta) = -\frac{2\mu}{\hbar^2 q} \int_0^\infty r dr \sin(qr) V(r)} \quad \begin{array}{l} \text{spherically} \\ \text{sym. potential} \end{array}$$

$$|\vec{q}|^2 = k^2 + k^2 - 2k^2 \cos\theta = 2k^2(1 - \cos\theta) = 4k^2 \sin^2 \frac{\theta}{2}$$

$$\boxed{q = 2k \sin \frac{\theta}{2}} \quad \begin{array}{l} \text{momentum transfer} \\ \text{in C.M. frame} \end{array}$$

Integral diverges for $V(r) \sim \frac{1}{r}$.

Coulomb potential "never lets go" to give plane wave.

Validity of Born Approximation

scattered wave is small

$$\frac{|\psi_{sc}(\vec{r} \rightarrow \infty)|}{|\psi_{in}(0)|} \ll 1$$

ψ_{sc} before asymptotic limit of $G_{\pm}$

$$|\psi_{sc}^{(0)}(0)| = \left| \frac{2\mu}{\hbar^2} \left(-\frac{1}{4\pi} \right) \int d^3r' \frac{e^{i\vec{k}\cdot\vec{r}'} V(r') e^{i\vec{k}\cdot\vec{r}'}}{r'} \right| \ll 1$$

general
criteria

$$= \left| \frac{2\mu}{\hbar^2 k} \int_0^\infty dr' V(r') e^{i\vec{k}\cdot\vec{r}'} \sin(kr') \right| \ll 1$$

High energy limit:

$$\frac{1}{2i} \left(e^{2i\vec{k}\cdot\vec{r}'} - 1 \right)$$

as $k \rightarrow \infty$, rapidly oscillating exponential integrator to zero. Criteria is then

$$\frac{\mu}{\hbar^2 k} \left| \int_0^\infty dr' V(r') \right| \ll 1$$

for sufficiently rapidly decreasing V , integral converges. Define $\int_0^\infty dr' V(r') \equiv V_0 r_0$

then

$$\boxed{\frac{\mu V_0 r_0^2}{\hbar^2} \ll k r_0}$$

high energy

expect Born to be valid for "large" k .

low energy limit examine criteria as $k \rightarrow 0$

$$\frac{2\mu}{\hbar^2 k} \left| \int_0^{\infty} dr V(r) e^{i kr} \sin(kr) \right| \ll 1$$

$$kr \rightarrow 0 \quad e^{i kr} \approx 1 \quad \sin(kr) \approx kr$$

$$\frac{2\mu}{\hbar^2} \left| \int_0^{\infty} r dr V(r) \right| \ll 1$$

for effective range r_0 , magnitude V_0

$$\boxed{\frac{\mu}{\hbar^2} V_0 r_0^2 \ll 1} \quad \text{low energy}$$

if this is satisfied, at high energy, $1 \ll kr_0$
and Born is valid for all energies.

Born Approximation Example

$$V(r) = g \frac{e^{-r/r_0}}{r} \quad r_0 = \frac{\hbar c}{m_0} \text{ range}$$

Yukawa potential

$$f^{(1)} = \frac{-2\mu}{\hbar^2 g} \int_0^\infty r dr \sin(qr) \left(\frac{g}{r}\right) e^{-r/r_0}$$

$$\sin qr = \frac{1}{2i} (e^{iqr} - e^{-iqr}) \quad \text{given}$$

$$f^{(1)}(q^2) = \frac{-2\mu g}{\hbar^2 (q^2 + m_0^2)}$$

$$q^2 = 4k^2 \sin^2 \theta/2$$

In limit $m_0 \rightarrow 0$ ($r_0 \rightarrow \infty$) amazing that this works!

$$f^{(1)}(\theta) = \frac{-2\mu g}{\hbar^2 4k^2 \sin^2 \theta/2}$$

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\theta)|^2 = \frac{\mu^2 g^2}{4\hbar^2} \frac{1}{k^4 \sin^4(\theta/2)}$$

For low energy α scattering (Rutherford) the LAB frame is the CM and $\hbar^2 k^2 = 2M_\alpha E$

$$g = Ze^2 Z$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4 Z^2}{4E^2} \sin^{-4}(\theta/2)$$

note: no $\hbar$ dependence; same as classical result

Addendum on Formal Scattering Theory

Lippman-Schwinger equation.

(refer to J.R. Taylor, Scattering Theory)

recall we wrote formal solution in terms of inverse operator,

$$\psi = \phi_{\vec{k}} + \frac{2\mu}{\hbar^2} (\nabla^2 + k^2 + i\varepsilon)^{-1} V \psi$$

now added $+i\varepsilon$ to operator to specify outgoing Green's function in contour integration.

This is in momentum space. In operator form,

with $\hat{H}_0 = \frac{\hbar^2}{2\mu} \nabla^2$ free particle
 $E = \frac{\hbar^2 k^2}{2\mu}$ incident energy

$$|\psi\rangle = |\phi\rangle + (E - \hat{H}_0 + i\varepsilon)^{-1} \hat{V} |\psi\rangle$$

Lippman-Schwinger. Relating back to Green's functions,

$$G_+(\vec{r}', \vec{r}'') = \frac{2\mu}{\hbar^2} \langle \vec{r}' | (E - \hat{H}_0 + i\varepsilon)^{-1} | \vec{r}'' \rangle$$

$$\tilde{G}_+(\vec{k}, \vec{k}'') = \langle \vec{k}' | (E - \hat{H}_0 + i\varepsilon)^{-1} | \vec{k}'' \rangle =$$

$$\left(\frac{2\mu}{\hbar^2} \right) \frac{(2\pi)^3 \delta^3(\vec{k}' - \vec{k}'')}{k^2 - k'^2 + i\varepsilon}$$

Define transition operator $\hat{T}$ by

$$\hat{T}|\phi\rangle \equiv \hat{V}|\psi\rangle \quad \text{or simply } |\phi_{\vec{k}}\rangle = |\vec{k}\rangle$$

$$\hat{V}|\psi\rangle = \hat{T}|\phi\rangle = \hat{V}|\phi\rangle + \hat{V} \frac{1}{E - \hat{H}_0 + i\epsilon} \hat{T}|\phi\rangle$$

Now we have operator equation,

$$\hat{T} = \hat{V} + \hat{V} (E - \hat{H}_0 + i\epsilon)^{-1} \hat{T}$$

with iterative solution

$$\hat{T} = \hat{V} + \hat{V} (E - \hat{H}_0 + i\epsilon)^{-1} \hat{V} + \hat{V} (E - \hat{H}_0 + i\epsilon)^{-1} \hat{V} (E - \hat{H}_0 + i\epsilon)^{-1} \hat{V} + \dots$$

Scattering amplitude

$$f(\theta, \phi) \equiv f(\vec{k}', \vec{k}) = \frac{-\mu}{2\pi\hbar^2} \int d^3r' e^{-i\vec{k}' \cdot \vec{r}'} V(\vec{r}') \psi(\vec{r}')$$

$$\text{with } \langle \vec{k}' | \vec{r}' \rangle = e^{-i\vec{k}' \cdot \vec{r}'}$$

$$\langle \vec{r}' | \psi \rangle = \psi(\vec{r}')$$

and $\langle \vec{r}' | \hat{V} = \langle \vec{r}' | V(\vec{r})$ so that

$$V(\vec{r}') \psi(\vec{r}') = \langle \vec{r}' | \hat{V} | \psi \rangle \text{ we get}$$

$$\begin{aligned} f(\vec{k}', \vec{k}) &= \frac{-\mu}{2\pi\hbar^2} \int d^3r' \langle \vec{k}' | \vec{r}' \rangle \langle \vec{r}' | \hat{V} | \psi \rangle \\ &= \frac{-\mu}{2\pi\hbar^2} \langle \vec{k}' | \hat{V} | \psi \rangle = \frac{-\mu}{2\pi\hbar^2} \langle \vec{k}' | \hat{T} | \vec{k} \rangle \end{aligned}$$

Then

$$f(\vec{k}', \vec{k}) = \sum_{n=1}^{\infty} f^{(n)}(\vec{k}', \vec{k}) \quad n \text{ is number of times } V \text{ appears}$$

$$f^{(1)} = -\frac{\mu}{2\pi\hbar^2} \langle \vec{k}' | \hat{V} | \vec{k} \rangle$$

$$f^{(2)} = -\frac{\mu}{2\pi\hbar^2} \langle \vec{k}' | \hat{V} (E - \hat{H}_0 + i\epsilon)^{-1} \hat{V} | \vec{k} \rangle$$

etc.

unpacking $f^{(2)}$:

$$f^{(2)} = -\frac{\mu}{2\pi\hbar^2} \langle \vec{k}' | \int d^3r' | \vec{r}' \rangle \langle \vec{r}' | \hat{V} (E - \hat{H}_0 + i\epsilon)^{-1} \int d^3r'' | \vec{r}'' \rangle$$

$$\langle \vec{r}'' | \hat{V} | \vec{k} \rangle$$

$$= -\frac{\mu}{2\pi\hbar^2} \int d^3r' \int d^3r'' \langle \vec{k}' | \vec{r}' \rangle V(\vec{r}') \langle \vec{r}' | (E - \hat{H}_0) | \vec{r}'' \rangle$$

$$\times V(\vec{r}'') \langle \vec{r}'' | \vec{k} \rangle$$

$$= -\frac{\mu}{2\pi\hbar^2} \int d^3r' \int d^3r'' e^{-i\vec{k}' \cdot \vec{r}'} V(\vec{r}') \left(\frac{2\mu}{\hbar^2} \right) G_4(\vec{r}', \vec{r}'') \times V(\vec{r}'') e^{+i\vec{k} \cdot \vec{r}''}$$