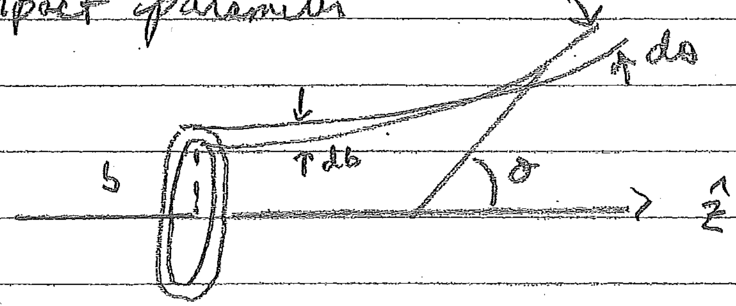
Lec 19: Scattering III

Partial waves -

classical scattering is described by impact parameter



$$\frac{d\sigma}{d\Omega}_{classical} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

for classical hard sphere, $\sigma = \pi a^2$

Classical impact parameter b is related to angular momentum

$$L = pb = \hbar k b \qquad \vec{L} \perp \hat{z}$$

For a finite range potential, $V(r) \approx 0$, $r > a$

maximum angular momentum that can contribute is

$$L^2_{max} = \hbar^2 (l)(l+1) = \hbar^2 k^2 a^2$$

$$l_{max}(l_{max}+1) = \hbar^2 k^2 a^2$$

as $k \to 0$, $l = 0$ (s-wave) dominates

<u>Method of partial waves</u> Spherically symmetric potential

Expand $f(\theta)$ in terms of spherical harmonics. For $V(r)$, $f(\theta)$ and only $Y_{\ell,0}(\theta)$ contribute.

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell,0}(\theta)$$ Legendre polynomials

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(k)\, P_\ell(\cos\theta)$$

p_0(x)=1; p_1(x)=x; p_2(x) = (1/2)(3x^2-1);…

and

$$\psi(\vec{r}) = \sum R_\ell(r) P_\ell(\cos\theta)$$

where $-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)^2 R_\ell + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} R_\ell + V R_\ell = E R_\ell$

In limit $kr \gg 1$, $V(r) \to 0$ and we have free particle solution.

Recall free particle solution for $V(r) = 0$:

$$R'' + \frac{2}{R}R' - \frac{\ell(\ell+1)}{r^2}R + k^2 R = 0$$

$k = \sqrt{\frac{2\mu E}{\hbar^2}}$

$$R_\ell(r) = j_\ell(kr),\ \eta_\ell(kr)$$

Now we cannot exclude η because we are finding the asymptotic (large r) solution.

$$j_\ell(kr) \sim \frac{\sin(kr - \ell\pi/2)}{kr}$$

$$\eta_\ell(kr) \sim -\frac{\cos(kr - \ell\pi/2)}{kr}$$

Therefore

$$\psi(r) \underset{kr \gg 1}{\sim} \sum_{l} \left[A_l \frac{\sin\theta_l}{kr} - B_l \frac{\cos\theta_l}{kr} \right] P_l(\cos\theta)$$

where $\theta_l = kr - l\pi/2$; This can be rewritten as

$$\psi(r) \underset{kr \gg 1}{\sim} \sum_{l} \frac{C_l}{kr} \sin\left(kr - l\pi/2 + \delta_l\right) P_l(\cos\theta)$$

where δ_l is the l^{th} partial wave phase shift.

To identify $f(\theta)$ we must subtract e^{ikz}:

$$e^{ikz} = e^{ikr\cos\theta} = \sum i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

$$\underset{kr \gg 1}{\sim} \sum i^l (2l+1) \left(\frac{1}{kr}\right) \sin\theta_l \; P_l(\cos\theta)$$

The coefficients C_l are determined by the requirement that in the asymptotic region, $\psi(r)$ contains only outgoing spherical waves.

Use $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$ to separate out the incoming & outgoing spherical waves.

scattered = total - incoming

$$\psi_{sc} \sim \sum_{l} \frac{P_l(\cos\theta)}{kr} D_l$$

$$D_l = \left(\frac{1}{2i}\right) C_l \left(e^{ikr} e^{-i\pi l/2} e^{i\delta_l} - e^{-ikr} e^{i\pi l/2} e^{-i\delta_l}\right)$$

$$- \left(\frac{1}{2i}\right) \underbrace{e^{i\pi l/2}}_{i^l} (2l+1)\left(\underbrace{e^{ikr} e^{-i\pi l/2}}_{\text{outgoing}} - \underbrace{e^{-ikr} e^{i\pi l/2}}_{\text{incoming}}\right)$$

incoming spherical wave must cancel:

so $$C_l = e^{i\delta_l}(2l+1)e^{i\pi l/2}$$

and $$D_l = e^{i\pi l/2}(2l+1)e^{ikr}e^{-i\pi l/2}\left(e^{2i\delta_l} - 1\right)$$

$$= (2l+1)\, e^{ikr} e^{i\delta_l} \sin\delta_l$$

$$\psi_{sc} \sim \frac{e^{ikr}}{r} f(\theta)$$

$$\boxed{f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta)\, e^{i\delta_l} \sin\delta_l}$$

δ_l contains all the physics of the scattering.

Total Cross Section: Orthogonality of Legendre polynomials

$$\int d\Omega \, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{4\pi}{\sqrt{(2\ell+1)(2\ell'+1)}} \int d\Omega \, Y_{\ell 0} Y_{\ell' 0}$$

$$= \frac{4\pi}{2\ell+1} \delta_{\ell\ell'}$$

$$\sigma_T = \int d\Omega \, |f(\theta)|^2 = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell$$

$$\equiv \sum_\ell \sigma_\ell$$

$$\sigma_\ell < \frac{4\pi}{k^2}(2\ell+1)$$ partial-wave unitarity

Optical Theorem: forward scattering wave destructively interferes with incident wave.

$$\mathrm{Im}\, f(0) = \frac{1}{k} \sum_\ell (2\ell+1) \sin^2\delta_\ell$$ p_l(1)=1

$$\boxed{\sigma_T = \frac{4\pi}{k} \mathrm{Im} f(0)}$$

Finding the phaseshift.

Solve $-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{d}{dr}r\right)^2 R_\ell + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} R_\ell + V(r) R_\ell$
$= E R_\ell$

Then take $r \to \infty$ and compare to:

$$C_\ell \frac{\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)}{kr}$$ asymptotic limit of full wave funct.

Example 1 Hard Sphere

$$V = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

for $\ell = 0$, let $U = Rr$

$$-\frac{\hbar^2}{2\mu} U'' = E U \qquad r > a$$

boundary condition $U(a) = 0$

$U = \sin, \cos$ or

$R = A\, j_0(kr) + B\, \eta_0(kr)$

where $j_0 = \frac{\sin kr}{kr}$, $\eta_0(kr) = -\frac{\cos kr}{kr}$

$R(a) = \frac{1}{ka}\left[A \sin ka - B \cos ka\right]$

$\frac{B}{A} = \tan(ka)$ satisfies boundary condition at r=a

$$R(r) = \frac{A}{kr}\left[\sin kr - \tan(ka)\cos kr\right]$$

here we already have the asymptotic form:

$$\frac{C_0}{kr}\sin(kr+\delta_0) = \frac{C_0}{kr}\left[\sin kr\cos\delta_0 + \cos kr\sin\delta_0\right]$$

$$= \frac{C_0\cos\delta_0}{kr}\left[\sin kr + \tan\delta_0\cos kr\right]$$

$$\boxed{\delta_0 = -ka}$$

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2(ka) \xrightarrow[ka\to 0]{} 4\pi a^2$$

which is 4 times the classical hard-sphere scattering.

S-wave phase shift & Scattering length

S-wave phase shift in low energy limit has simple, physical interpretation.

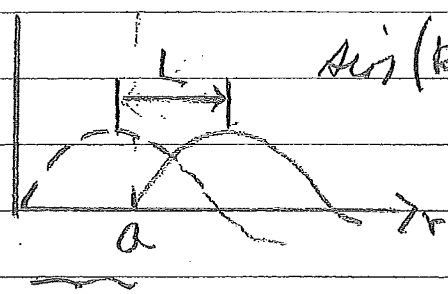
define scattering length $L \equiv -\lim_{k\to 0} f(\theta)$

at low energy, $f(\theta) \cong \frac{e^{i\delta_0}}{k}\sin\delta_0$
$= \frac{1}{k}(\sin\delta_0\cos\delta_0 + i\sin^2\delta_0)$

for $\delta_0 \ll 1$, $f(\theta) \approx \frac{\delta_0}{k}$

then $\delta_0 = -kL$

For hard sphere, $\delta_0 = -ka$ and $L = a$.



$\sin(kr+\delta_0) = \sin(kr-ka)$

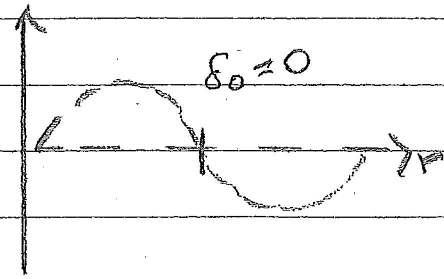
$= \sin\left[\frac{2\pi}{\lambda}(r-a)\right]$

$L = \frac{-\delta_0}{k} = a$

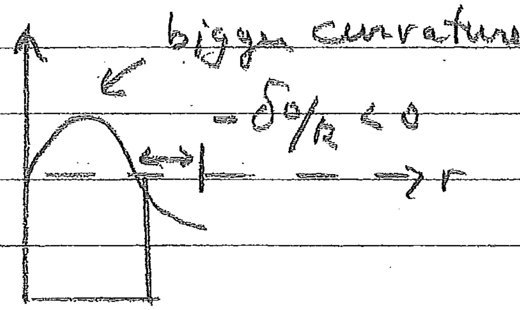
repulsive potential gives + scattering length

Sketch of scattering lengths

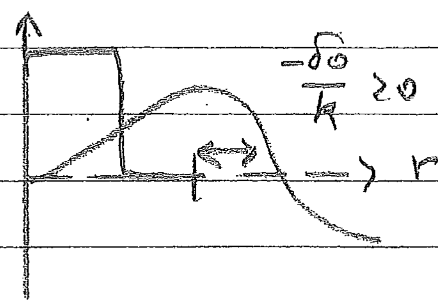
No potential
$V(r)=0$



attractive
$V(r)$



repulsive
$V(r)$



Example: spherical δ-function

$$\frac{2m}{\hbar^2} V(r) = \gamma \delta(r-a)$$

S-wave scattering: let U=Rr

$$U'' + k^2 U = \gamma \delta(r-a) U$$

satisfying boundary at r=0,

$$U = \sin kr \qquad r<a$$
$$U_> = B \sin(kr+\delta_0) \qquad r>a$$

at r=a: $\sin ka = B \sin(ka+\delta_0)$

integrating over $a \pm \varepsilon$

$$U'|_{a+} - U'_{a-} = \gamma U|_a$$

$$B k \cos(ka+\delta_0) - k\cos ka = \gamma \sin ka$$

$$\tan(ka+\delta_0) = \frac{\tan(ka)}{1+\frac{\gamma}{k}\tan(ka)}$$

define $t = \tan ka$ tan(ka+δ0)=

$$\frac{t+\tan\delta_0}{1-t\tan\delta_0} = \frac{t}{1+\frac{\gamma}{k}t}$$

$$t' = \tan(ka+\delta_0) = \frac{t+\tan\delta_0}{1-t\tan\delta_0} = \frac{t}{1+\frac{\gamma}{k}t}$$

solve for δ_0 and then $\sigma = \frac{4\pi}{k^2}\sin^2\delta_0$

At low energy, $ka \ll 1$ then $t \approx ka$

$$\frac{ka+t_0}{1-ka\,t_0} = \frac{ka}{1+\gamma a}$$

$$(ka+t_0) = \frac{ka}{1+\gamma a}(1-ka\,t_0)$$

$$t_0\underbrace{\left[1+\frac{(ka)^2}{1+\gamma a}\right]}_{\approx 1} = \frac{ka}{1+\gamma a} - ka = \frac{-\gamma a\,ka}{1+\gamma a}$$

$$t_0 = \tan\delta_0 = -ka\left(\frac{\gamma a}{1+\gamma a}\right)$$

so $\sin\delta_0 \approx \tan\delta_0 = -ka\left(\frac{\gamma a}{1+\gamma a}\right)$

$$\sigma_0 = 4\pi a^2\left(\frac{\gamma a}{1+\gamma a}\right)^2$$

compare to $\sigma_B = 4\pi a^4\gamma^2$

Born is valid for ~~~Lec. 18, page 4 "general criteria"

$$\left|\frac{1}{k}\int_0^\infty dr\, e^{ikr} \sin kr\, \gamma\,\delta(r-a)\right| << 1$$

$$\frac{1}{k^2}\left|\int_0^\infty dx\, e^{ix} \sin x\, \delta\left(\frac{x}{k}-a\right)\right| << 1$$

* $$\left(\frac{1}{k^2}\right)\left(\frac{1}{k}\right)^{-1}\gamma\left|e^{ika} \sin ka\right| << 1$$

$\sin ka \simeq ka$ for $ka << 1$

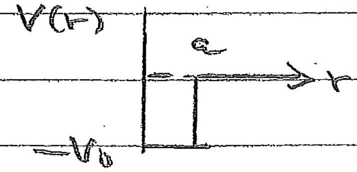
Born is valid for $\gamma a << 1$

σ_0 for $\gamma a << 1$ is $\sigma_0 \simeq 4\pi a^2 (\gamma a)^2 = \sigma_B$.

partial wave result in this limit

* note $\displaystyle\int f(x)\delta(ax)\,dx = \frac{f(0)}{|a|}$

Finite Well Scattering: RT effect, resonances



$$V(r) = \begin{cases} -V_0 & r<a \\ 0 & r>a \end{cases}$$

$l=0$ equation for $U(r) = R(r)\,r$:

$$U'' + k_0^2 = 0 \quad r<a \quad k_0 = \sqrt{2\mu(E+V_0)}/\hbar$$
$$U'' + k^2 = 0 \quad r>a \quad k = \sqrt{2\mu E}/\hbar$$

solution:

$$U_< = \sin k_0 r$$
$$U_> = C \sin(kr + \delta_0)$$

matching condition @ boundary $r=a$

$$\tan(ka + \delta_0) = \left(\frac{ka}{k_0 a}\right)\tan(k_0 a)$$

Case 1: $ka \ll 1$, $k_0 a \neq \pi/2$, δ_0 small

$$ka + \delta_0 = \frac{ka}{k_0 a}\tan k_0 a$$
$$\delta_0 = ka\left(\frac{\tan k_0 a}{k_0 a} - 1\right) \approx \sin\delta_0$$

$$\sigma = 4\pi a^2\left[\frac{\tan k_0 a}{k_0 a} - 1\right]^2$$

$\lim ka \to 0$, $k_0 a = a\sqrt{2\mu V_0}/\hbar$ and

σ independent of k.

Case 2: $k_0 a \simeq \pi/2$, $ka \ll 1$

$$\tan(ka+\delta_0) \approx \tan\delta_0 = \frac{k}{k_0}\tan(k_0 a) \to \infty$$

$\delta_0 = \pi/2$. Expect $l=0$ to dominate for $ka \ll 1$

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}$$ saturates unitarity bound

This effect is called resonance scattering and is due to the existence of a low energy bound state

$$(k_0 a)^2 - (ka)^2 = \frac{2\mu V_0 a^2}{\hbar^2} \qquad \text{scattering}$$

$k \to iq \qquad (k_0 a)^2 + (qa)^2 = \frac{2m V_0 a^2}{\hbar^2} \geq \frac{\pi}{2}$ bound state

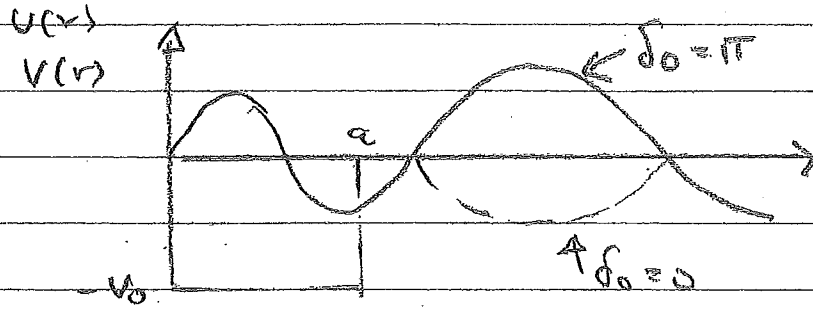
condition for existence of finite well bound state

If potential has bound state near $E=0$ ($q=0$) σ_0 has a maximum near $k=0$.

Case 3: $\delta_0 = \pi$

For attractive potential it is possible to get $\delta_0 = \pi$.

$$u_\ell = C \sin(kr + \pi) = -C \sin kr$$



Sketch of $\delta_0 = \pi$ wave function

If k is sufficiently low that $\sigma \approx \sigma_0$ scattering is near zero! "transmission resonance" (Ramsauer-Townsend effect) observed for low energy ($E \sim 0.7eV$) electron-atom scattering by Noble gases (Ne, Ar, ...).

Resonance Scattering

Near short-lived ("quasi") bound states, cross section exhibits an enhancement called a resonance.

phase shift is function of k or equivalently E:

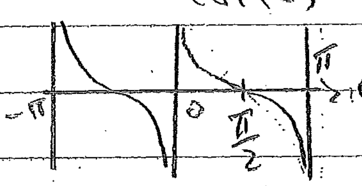
$$\delta_\ell(E_0) = \pi/2 \quad \text{resonance}$$

Near resonance, parametrize by first term in Taylor expansion:

$$f_\ell = (2\ell+1)\, a_\ell(k)\, P_\ell(\cos\theta)$$

$$= (2\ell+1)\, \frac{e^{i\delta_\ell}}{k} \sin\delta_\ell\, P_\ell(\cos\theta)$$

$$a_\ell(k) = \frac{e^{i\delta_\ell}}{k} \sin\delta_\ell = \frac{1}{k}\, \frac{1}{\cot\delta_\ell - i}$$



$$\cot\delta_\ell \cong \underbrace{\cot\delta_\ell(E_0)}_{0} + \underbrace{\frac{d\cot\delta_\ell}{dE}}_{-\frac{1}{\sin^2\delta_\ell}\frac{d\delta_\ell}{dE}\Big|_{E=E_0} = -\frac{d\delta_\ell}{dE}\Big|_{E=E_0} \equiv -\frac{2}{\Gamma}} (E - E_0)$$

$$\boxed{\cot\delta_\ell(E) \cong -\frac{2}{\Gamma}(E - E_0)}$$

parameterization in E_0, Γ becomes

$$a_\ell(k) \cong \frac{1}{k}\left(\frac{-2}{\Gamma}(E-E_0)-i\right)^{-1}$$

$$= \frac{-1}{k}\,\frac{\Gamma/2}{(E-E_0)+i\Gamma/2}$$

$$\sigma_\ell = \int d\Omega\,|f|^2 = \sum_\ell 4\pi(2\ell+1)\,|a_\ell(k)|^2$$

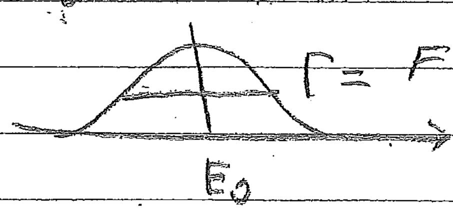
$$\int d\Omega\, P_\ell P_{\ell'} = \left(\frac{4\pi}{2\ell+1}\right)\delta_{\ell\ell'}$$

total is then sum of total from each partial wave

Kronecker-delta

$$\sigma_\ell = \frac{4\pi(2\ell+1)}{k^2}\left(\frac{\Gamma^2/4}{(E-E_0)^2+(\Gamma/2)^2}\right)$$

Breit-Wigner resonance shape.
long-tailed Gaussian



Total elastic σ for spinless particle
to produce resonance of spin J, $2\ell+1 \to 2J+1$

$$p+\gamma \to \Delta^+(J=3/2) \to \begin{Bmatrix} p\pi^0 \\ N\pi^+ \end{Bmatrix} \quad 99.4\% \text{ Branching}$$

$$p\gamma \quad 0.6\%$$

$M_\Delta = 1236$ MeV, $\Gamma = 115$ MeV

$\sigma(E_0) \simeq 200$ mB

GZK effect $p+\gamma \to \Delta^+ \to \begin{cases} p\pi^0 \\ n\pi^+ \end{cases}$

Ultra-high energy cosmic ray interacting with 2.7°K photons from CMB.

$$\langle E_\gamma \rangle_{CMB} \approx 10^{-4}\ \text{eV}$$

$$E_p^{cm} = m_{\Delta^+}/2 \qquad m_{\Delta^+} = 1232\ \text{MeV}/c^2$$

$$E_p \approx \frac{1}{8}\,\frac{m_\Delta^2}{E_\gamma} \approx 10^{21}\ \text{eV}$$

taking into account CMB photon distribution,

$$E_p^{th} \sim 5\times10^{19}\ \text{eV}$$

Expect cut-off in spectrum of cosmic rays if sources are $\gtrsim$ 50 mpc.

Evidence from recent measurements
HiRes, Auger