Lec 20: EM potential

So far we have only considered <u>static</u> $\vec{E}, \vec{B}$ fields (except for simple spin-resonance example).

<u>Goals</u> ① time-varying fields & hence time-dependent perturbation theory
② spontaneous emission
typical excited atomic state lifetime $\mathcal{O}(\text{ns})$

How do we include $\vec{E}(t), \vec{B}(t)$ into Q.M. Hamiltonian? Recall Lorentz force

$$\vec{F} = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

But H depends on energy, so we must use the potentials.

For <u>static</u> fields; $\vec{E} = -\nabla\varphi$, $V = q\varphi$

$$\vec{F} = -\vec{\nabla}V = q(-\vec{\nabla}\varphi) = q\vec{E}$$

and $\quad \vec{B} = \vec{\nabla} \times \vec{A}$

Maxwell's Equations

① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ ② $\vec{\nabla}\cdot\vec{B} = 0$

③ $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ ④ $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

$\vec{B} = \vec{\nabla}\times\vec{A}$ automatically satisfies ②
because $(\vec{\nabla}\cdot\vec{\nabla}\times)\vec{A} \equiv 0$

then $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ satisfies ③:

$$\vec{\nabla}\times\vec{E} = \underbrace{-\vec{\nabla}\times\vec{\nabla}\varphi}_{\text{identically zero}} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

①, ④ become:

$$\nabla^2\varphi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\nabla^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\varphi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = -\frac{4\pi}{c}\vec{J}$$

Note: in 4-vector notation $x^\mu \equiv (ct, \vec{x})$
$\underset{\sim}{x}\cdot\underset{\sim}{x} = (ct)^2 - \vec{x}\cdot\vec{x}$

$A^\mu \equiv (\varphi, \vec{A})$ $j^\mu \equiv (c\rho, \vec{J})$

$$\partial_\mu \underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{\equiv F^{\mu\nu}} = \frac{4\pi}{c} j^\nu$$

Gauge Invariance in E+M

Potentials are not unique.

Maxwell's equations are invariant under

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \quad ; \quad \varphi \to \varphi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

where $\Lambda(\vec{x}, t)$ is an arbitrary function.
In 4-vector notation:

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$$A^\mu \to A^\mu - \partial^\mu \Lambda$$

Invariance of Maxwell's equations is obvious in 4-vector notation.

Hamiltonian (Appendix E)

Classical Lagrangian:

$$L = \frac{1}{2} m v^2 - q\varphi + \frac{q}{c}\vec{A}\cdot\vec{v}$$

Lagrange equations, $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$ gives

classical equation of motion $\quad p_i \equiv \frac{\partial L}{\partial \dot{x}_i} \quad \vec{p} = m\vec{v} + \frac{q}{c}\vec{A}$

$$m\ddot{\vec{x}} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

Corresponding H is:

$$H = \frac{(\vec{p} - q\vec{A}/c)^2}{2m} + q\varphi + \text{free field energy}$$

Gauge Invariance in Q.M.

$$\hat{H} = (\vec{p} - \tfrac{q}{c}\vec{A})^2/2m + q\varphi \quad ; \quad \vec{p} \to \tfrac{\hbar}{i}\vec{\nabla}$$

$$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi$$

$$\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)\psi = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2\psi$$

Schrödinger eq. invariant under transformation
(where $\Lambda(\vec{x},t)$ is arbitrary function)

$$\psi \to \psi' = e^{iq\Lambda/\hbar c}\,\psi \qquad \text{local phase}$$
$$\varphi \to \varphi' = \varphi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$
$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

To show, use $\left[i\hbar\frac{\partial}{\partial t}, e^{iq\Lambda/\hbar c}\right] = -\frac{q}{c}e^{iq\Lambda/\hbar c}\frac{\partial\Lambda}{\partial t}$

then $\left(i\hbar\frac{\partial}{\partial t} - q\varphi'\right)\psi' = \left(i\hbar\frac{\partial}{\partial t} - q\varphi + \frac{q}{c}\frac{\partial\Lambda}{\partial t}\right)e^{iq\Lambda/\hbar c}\psi$

$$= e^{iq\Lambda/\hbar c}\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)\psi$$

And $\left[\vec{p}, e^{iq\Lambda/\hbar c}\right] = \frac{q}{c}\vec{\nabla}\Lambda\, e^{iq\Lambda/\hbar c}$

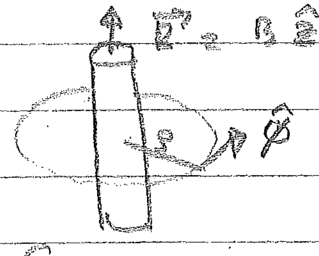
$$\left(\vec{p} - \frac{q}{c}\vec{A}'\right)e^{iq\Lambda/\hbar c} = \left(\vec{p} - \frac{q}{c}\vec{A} - \frac{q}{c}\vec{\nabla}\Lambda\right)e^{iq\Lambda/\hbar c}$$
$$= e^{iq\Lambda/\hbar c}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$$

So $\quad \left(\vec{p} - \frac{q}{c}\vec{A}'\right)^2\psi' = e^{iq\Lambda/\hbar c}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2\psi$

Does $\vec{A}$ exist physically?

<u>Aharonov - Bohm effect</u>

$\vec{A}$ can be nonzero where $\vec{B}=0$ Consider very long solenoid:



$\vec{A} = A\hat{\phi}$

$$\int \vec{B}\cdot d\vec{S} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{r}$$

Let a be radius of solenoid,

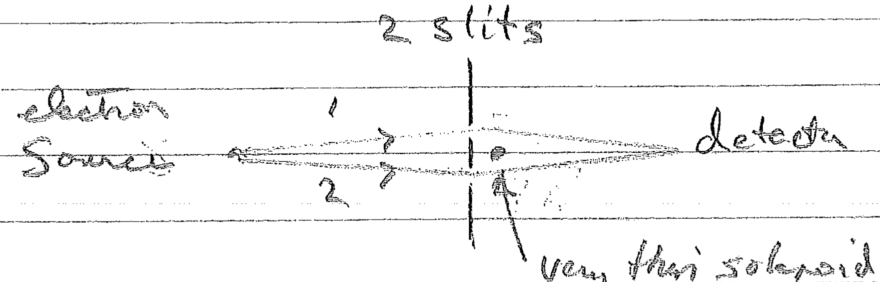
$$\rho < a \qquad B\cdot\pi\rho^2 = A\cdot 2\pi\rho \;, \quad \vec{A} = \left(\frac{B\rho}{2}\right)\hat{\phi}$$

$$\rho > a \qquad B\pi a^2 = A\, 2\pi\rho \;, \quad \vec{A} = \frac{Ba^2}{2\rho}\hat{\phi}$$

Check that $\vec{A}$ is continuous at $\rho = a$ and

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{z}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\rho\right)A = \begin{cases} 0 & \rho > a \\ B\hat{z} & \rho < a \end{cases}$$

<u>Experiment</u>: 2 path interference



e^- paths interfere

Semiclassical approximation of wave function:

$$\psi_1(d) = \psi(s) \exp\left[\frac{i}{\hbar}\int_1 L\,dt\right]$$

$$\int L\,dt = \frac{q}{c}\int \vec{A}\cdot\vec{v}\,dt = \int \vec{A}\cdot dx$$

$$\psi_1(d) = \psi_1(s) \exp\Big[\underbrace{\frac{iq}{\hbar c}\int \vec{A}\cdot dx}_{\equiv i\phi_1}\Big]$$

Paths 1 & 2 interfere:

$$\psi_{tot} = \psi_1(d) + \psi_2(d) = \psi(s)\left(e^{i\phi_1} + e^{i\phi_2}\right)$$

$$= \psi(s)\, e^{\frac{i\phi_1}{2}} e^{\frac{i\phi_2}{2}} \left[e^{i(\phi_1-\phi_2)/2} + e^{-i(\phi_1-\phi_2)/2}\right]$$

$$\phi_1 - \phi_2 \equiv \Delta\phi = \frac{q}{\hbar c}\left[\int_1 \vec{A}\cdot dx - \int_2 \vec{A}\cdot dx\right]$$

$$= \frac{q}{\hbar c}\oint \vec{A}\cdot d\vec{x} = \frac{q}{\hbar c}\int \vec{B}\cdot d\vec{s} = \frac{q}{\hbar c}\Phi$$

$$|\psi_{tot}|^2 \propto \cos\left(\frac{q}{\hbar c}\Phi\right)$$

interference depends on gauge-invariant flux Φ.

Phase shift observed even though particle never enters $B \neq 0$ region.