Lec 21: The Photon

The photon is the quantum of the field $\vec{A}$. Maxwell in Coulomb gauge $(\vec{\nabla}\cdot\vec{A}=0)$

$$\nabla^2\varphi + \frac{1}{c}\frac{\partial}{\partial t}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}\right) = -4\pi\rho$$

$$\nabla^2\vec{A} - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\varphi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) - \vec{\nabla}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}\right) = -\frac{4\pi}{c}\vec{J}$$

<u>Note</u>: not <u>manifestly</u> Lorentz Invariant

$$\varphi(\vec{r},t) = \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}$$

connects φ to ρ at same instant in time

<u>Free field equations</u> if $\rho=0$, $\vec{j}=0$ everywhere
$\varphi=0$ and

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\vec{A}=0$$

plane wave solution $\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$
$\omega = c|\vec{k}|$. Gauge condition

$$\vec{\nabla}\cdot\vec{A} = 0 = \vec{k}\cdot\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

implies $\vec{A}\perp\vec{k}$

also called transverse gauge

Photon has only two polarization states

$$|x\rangle = \frac{1}{\sqrt{2}}\left(|R\rangle + |L\rangle\right)$$
$$|y\rangle = \frac{1}{i\sqrt{2}}\left(|R\rangle - |L\rangle\right)$$

$|R\rangle = |1,+1\rangle$; $|L\rangle = |1,-1\rangle$

Longitudinal polarization $|1,0\rangle$ forbidden by gauge invariance. Such a term would appear for massive photon adding to $\mathcal{L} \sim \frac{1}{2}m^2A^2$ explicitly violating gauge invariance.

<u>Free field Classical H.</u>

$$H_{EM} = \frac{1}{8\pi}\int d^3r \left(|\vec{E}|^2 + |\vec{B}|^2\right)$$
$$= \frac{1}{8\pi}\int d^3r \left[\frac{1}{c^2}\left|\frac{\partial \vec{A}}{\partial t}\right|^2 + \left|\vec{\nabla}\times\vec{A}\right|^2\right]$$

Coulomb gauge

Fourier transform $\vec{A}(\vec{r},t) = \int d^3k \, \vec{A}_{\vec{k}} \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

quantum theory quantizes each field mode (degree of freedom).

<u>Box Normalization</u> For ease of counting modes quantize in box of volume $V = L^3$. then let $L \to \infty$

$$e^{ik_x x} = e^{ik_x(x+L)} \qquad k_x = \frac{2\pi n_x}{L} \qquad n_x = \pm 1, \pm 2, \ldots$$

$$\vec{A}_k = C_{\vec{k},\lambda}\,\vec{\varepsilon}(\vec{k},\lambda) \qquad \vec{\varepsilon} \text{ unit polarization vector}$$

where $\vec{k} = \hat{x}\left(\frac{2\pi n_x}{L}\right) + \hat{y}\left(\frac{2\pi n_y}{L}\right) + \hat{z}\left(\frac{2\pi n_z}{L}\right)$

and $\vec{k}\cdot\vec{\varepsilon} = 0$

General solution of wave equation. $\vec{A}$ is real field and take $n_x > 0$, $n_y > 0$, $n_z > 0$.

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\substack{\vec{k} \\ n_x, n_y, n_z}} \sum_{\lambda=1}^{2} \left[C_{k\lambda}\, \vec{\varepsilon}(k,\lambda)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + C^*_{k\lambda}\, \vec{\varepsilon}^{\,*}(k,\lambda)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

modes are orthonormal:

$$\int_{-L/2}^{L/2} dx \left(\frac{1}{\sqrt{L}} e^{-im\frac{2\pi x}{L}}\right)\left(\frac{1}{\sqrt{L}} e^{in\frac{2\pi x}{L}}\right) = \frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{i(n-m)\frac{2\pi x}{L}}$$

$$= \begin{cases} \frac{1}{L}\, \frac{L}{(n-m)\pi} \sin\left[(n-m)\pi\right] = 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$= \delta_{nm} \xrightarrow[L\to\infty]{} \delta(k_x - k_x')$$

in 3 dimensions write as $\delta_{\vec{k}\vec{k}'}$

H in terms of plane waves:

use

$$\frac{1}{c}\frac{\partial}{\partial t} C_{\vec{k}\lambda}\vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{-i\omega}{c} C_{\vec{k}\lambda}\vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

and

$$\vec{\nabla}\times\left(C_{\vec{k}\lambda}\vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) = i\vec{k}\times\vec{\varepsilon}\, C_{\vec{k}\lambda}\vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

plus orthogonality to get

$$\boxed{H_{EM} = \frac{1}{2\pi}\sum_{\vec{k},\lambda} k^2 \left|C_{\vec{k}\lambda}\right|^2} \qquad |\vec{k}| = \frac{\omega}{c}$$

<u>Free field quantization</u> What is analog of $[\hat{x},\hat{p}] = i\hbar$?

Define time dependent amplitude

$$C_{\vec{k},\lambda}(t) \equiv C_{\vec{k}\lambda}(0)\, e^{-i\omega t}$$

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}}\sum_{\vec{k},\lambda}\left[C_{\vec{k}\lambda}(t)\,\vec{\varepsilon}\, e^{i\vec{k}\cdot\vec{r}} + C^*_{\vec{k}\lambda}(t)\,\vec{\varepsilon}^{\,*}\, e^{-i\vec{k}\cdot\vec{r}}\right]$$

C's satisfy $\frac{d^2}{dt^2} C_{\vec{k},\lambda} = -\omega^2 C_{\vec{k},\lambda}$

define

$$q_{\vec{k},\lambda}(t) \equiv \frac{1}{c\sqrt{4\pi}}\left(C_{\vec{k},\lambda} + C^*_{\vec{k},\lambda}\right)$$

$$p_{\vec{k},\lambda}(t) = \frac{-i\omega}{c\sqrt{4\pi}}\left(C_{\vec{k},\lambda} - C^*_{\vec{k},\lambda}\right)$$

then

$$\dot{q}_{\vec{k},\lambda} = p_{\vec{k},\lambda}$$

$$\dot{p}_{\vec{k},\lambda} = -\omega^2 q_{\vec{k}\lambda}$$

$$H_{EM} = \frac{1}{2}\sum_{\vec{k},\lambda}\left[P^2_{\vec{k},\lambda} + \omega^2 q^2_{\vec{k},\lambda}\right]$$

q, p are canonically conjugate

$$\frac{\partial H}{\partial q} = \omega^2 q = -\dot{p}$$

$$\frac{\partial H}{\partial p} = \dot{q}$$

Hamilton's equations

Quantize with <u>equal-time</u> commutators

$$\left[\hat{q}_{\vec{k},\lambda}(t), \hat{P}_{\vec{k}'\lambda'}(t)\right] = i\hbar\, \delta_{\vec{k},\vec{k}'}\, \delta_{\lambda\lambda'}$$

Define creation, annihilation operators

recall $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) = \sqrt{\frac{\omega}{2\hbar}}\left(\hat{x}\sqrt{m} + \frac{i}{\omega}\frac{\hat{p}}{\sqrt{m}}\right)$

so $\hat{a}_{\vec{k},\lambda} \equiv \frac{\omega}{2\hbar}\left(\hat{q}_{\vec{k}\lambda} + \frac{i}{\omega}\hat{P}_{\vec{k}\lambda}\right)$

Compare to previous expansion

$$C_{\vec{k},\lambda} \longrightarrow \sqrt{\frac{2\pi\hbar}{\omega}}\, \hat{a}_{\vec{k},\lambda}$$

Quantum field $\hat{\vec{A}}$:

$$\vec{A}(\vec{r},t) = \frac{c}{\sqrt{V}} \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar}{\omega}} \left(\hat{a}(t)\, \vec{\varepsilon}\, e^{i\vec{k}\cdot\vec{r}} + \hat{a}^{+}(t)\, \vec{\varepsilon}^{\,*} e^{-i\vec{k}\cdot\vec{r}} \right)$$

where $\left[\hat{a}(t), \hat{a}^{+}(t)\right] = \delta_{\vec{k}\vec{k}'}\,\delta_{\lambda\lambda'}$

$$\hat{H} = \sum_{k,\lambda} \hbar\omega \left(\underbrace{\tfrac{1}{2}}_{\text{vacuum energy}} + \hat{a}^{+}_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda} \right)$$

Number operator $\hat{a}^{+}\hat{a}$ is independent of time because $\hat{a}(t) = \hat{a}(0)\, e^{-i\omega t}$

<u>Photon states</u> $\quad \hat{a}_{\vec{k}\lambda} |0\rangle = 0$

$$\hat{H}|0\rangle = \frac{1}{2} \sum_{\vec{k},\lambda} \hbar\omega |0\rangle = \underbrace{E_0}_{\infty\ \text{constant}} |0\rangle$$

$$|\vec{k},\lambda\rangle = \hat{a}^{+}_{\vec{k},\lambda} |0\rangle$$

$$|n_{\vec{k}},\lambda\rangle = \frac{(a^{+}_{\vec{k},\lambda})^{N}}{\sqrt{N!}} |0\rangle \qquad n\text{-photon state}$$

$$\hat{\vec{p}} = \frac{1}{4\pi c} \int d^3r\, \vec{E}\times\vec{B} = \sum_{k,\lambda} \hbar\vec{k}\, \hat{a}^{+}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{p}}\, |n_{\vec{k}},\lambda\rangle = \hbar\vec{k}\, n_{\vec{k}}\, |n_{\vec{k}},\lambda\rangle$$

photons carry momentum (and energy)

L, R polarization states carry angular momentum

$$|1,\pm 1\rangle = \frac{1}{\sqrt{2}}\left[|\vec{k},1\rangle \pm i\,|\vec{k},2\rangle\right] \qquad \begin{array}{l} 1 \equiv \hat{x},\ 2 \equiv \hat{y} \\ \vec{k} = k\hat{z} \end{array}$$

$$= \frac{1}{\sqrt{2}}\underbrace{\left(\hat{a}^{+}_{\vec{k},1} \pm i\,\hat{a}^{+}_{\vec{k},2}\right)}_{\equiv\ \hat{a}^{+}_{\vec{k},\pm}}|0\rangle$$

$$\vec{J} = \int d^3r\ \vec{r} \times \left(\frac{\vec{E}\times\vec{B}}{4\pi c}\right)$$

can show $\dfrac{\hat{\vec{J}}\cdot\vec{k}}{k}\,|1,\pm 1\rangle = \pm\hbar\,|1,\pm 1\rangle$

Hamiltonian for atom + field

$$\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{r} + \frac{1}{8\pi}\int d^3r \left(\left(\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)^2 + \left(\vec{\nabla}\times\vec{A}\right)^2 \right)$$

with rest of $\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$ going into

$$\hat{H}_1 = \frac{e}{2mc}\vec{A}\cdot\left(\frac{\hbar}{i}\vec{\nabla}\right) + \frac{e}{2mc}\frac{\hbar}{i}\left(\vec{\nabla}\cdot\vec{A}\right) + \frac{e^2}{2mc^2}\left|\vec{A}\right|^2$$

$$\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0 \text{ in Coulomb gauge}} + \vec{A}\cdot\vec{\nabla}$$

$$\hat{H}_1(t) = \frac{e}{mc}\vec{A}(t)\cdot\left(\frac{\hbar}{i}\vec{\nabla}\right) + \frac{e^2}{2mc^2}\left|\vec{A}(t)\right|^2$$

Need time-dependent perturbation theory