Lecture 23: Atomic Transitions

Spontaneous emission: atom interacts with vacuum.

$$|i\rangle = |\psi_i\rangle|0\rangle$$
$$|f\rangle = |\psi_f\rangle|\vec{k},\lambda\rangle$$

Contribution from $\hat{H}_1$ w

$$\hat{H}_1 = \frac{e}{mc}\hat{\vec{A}}^{+}\cdot\hat{\vec{p}} \quad , \quad \hat{\vec{A}}^{+} = \frac{c}{\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\,\hat{a}^{+}_{\vec{k},\lambda}\,\vec{\varepsilon}^{\,*}\,e^{-i\vec{k}\cdot\vec{r}}$$

$$k^{-1} = \frac{\hbar c}{E_\gamma} \gtrsim \frac{200\,\text{eV}\cdot\text{nm}}{10\,\text{eV}} = 20\,\text{nm}$$

$$k a_0 = \frac{0.05\,\text{nm}}{20\,\text{nm}} \ll 1$$

<u>dipole approximation</u> $\quad e^{-i\vec{k}\cdot\vec{r}} \approx 1$

trick to evaluate $\langle\hat{\vec{p}}\rangle$

$$[\hat{H}_0, \hat{r}_i] = \left[\frac{\hat{p}^2}{2m} - \frac{e^2}{r}, \hat{r}_i\right] = \left[\frac{\hat{p}^2}{2m}, \hat{r}_i\right] = -\frac{i\hbar}{m}\hat{p}_i$$

so $\quad \hat{\vec{p}} = \frac{im}{\hbar}[\hat{H}_0, \hat{\vec{r}}]$

photon part $\quad \langle\vec{k},\lambda|\hat{a}^{+}_{\vec{k},\lambda}|0\rangle = 1$

atomic part:

$$\vec{\varepsilon}^{\,*}_{\vec{k}\lambda} \cdot \langle \psi_f | \hat{\vec{p}} | \psi_i \rangle = \frac{im}{\hbar} \vec{\varepsilon}^{\,*} \langle \psi_f | [\hat{H}_0, \hat{\vec{r}}] | \psi_i \rangle$$

$$= \frac{im}{\hbar} E_\gamma \langle \psi_f | \underbrace{\vec{\varepsilon}^{\,*} \cdot \hat{\vec{r}}}_{\text{electric dipole}} | \psi_i \rangle$$

$E_\gamma = E_f - E_i$

$$|\langle f | H_i | i \rangle|^2 = \underbrace{\left(\frac{e}{mc}\right)^2 \frac{c^2}{V} \left(\frac{2\pi\hbar}{\omega}\right) \frac{m^2}{\hbar^2} E_\gamma^2}_{\frac{\alpha\hbar c}{c^2}\left(\frac{c^2}{V}\right)\left(\frac{2\pi\hbar^2}{E_\gamma}\right)\frac{E_\gamma^2}{\hbar^2} = \frac{2\pi}{V}\alpha\hbar c E_\gamma} \left| \langle \psi_f | \vec{\varepsilon}^{\,*} \cdot \hat{\vec{r}} | \psi_i \rangle \right|^2$$

Rate is:

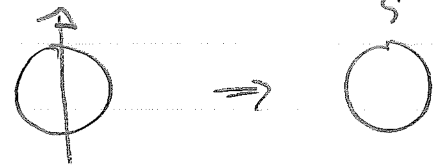
$$\frac{dR}{d\Omega_{\vec{k}}} = \frac{2\pi}{\hbar} \underbrace{\left[\frac{V}{(2\pi)^3} \frac{E_\gamma^2}{(\hbar c)^3} \right]}_{\rho(E_\gamma)} \frac{2\pi}{V} \alpha\hbar c E_\gamma \sum_\lambda \left| \langle \psi_f | \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} \cdot \hat{\vec{r}} | \psi_i \rangle \right|^2$$

$$= \frac{1}{2\pi} \frac{\alpha}{\hbar^3 c^2} E_\gamma^3 \sum_\lambda \left| \langle \psi_f | \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} \cdot \hat{\vec{r}} | \psi_i \rangle \right|^2$$

check dimensions $\frac{[E]^3}{[E \cdot \ell]^3} c [\ell]^2 = \frac{1}{time}$

Consider $|2p\rangle = |2,1,1\rangle \Longrightarrow |1s\rangle$

$\vec{J} = \hbar \hat{z}$ $\qquad\qquad$ γ $\vec{J} = \hbar \hat{z}$



$J = 0$

right-circular polarized γ emitted along $\hat{z}$-axis

$$\frac{dR_{+z}}{d\Omega} = \frac{1}{2\pi} \frac{\alpha E_\gamma^3}{c^2 \hbar^3} \left| \langle 1,0,0 | \vec{\varepsilon}^{\,*} \cdot \vec{r} \, | 2,1,1 \rangle \right|^2$$

*

$$\vec{\varepsilon}_{\pm} = \frac{1}{\sqrt{2}} (\hat{x} \pm i \hat{y}) \qquad \text{recall } Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \frac{(x \pm i y)}{r}$$

$$\vec{r} \cdot \vec{\varepsilon}_{+} = \frac{1}{\sqrt{2}} (x + i y) = -\sqrt{\frac{4\pi}{3}} \, r \, Y_{1+1} \; \Big\rangle^{*}$$

$$| \; |^2 = |\langle r \rangle|^2 \frac{4\pi}{3} \left| \int d\Omega \, Y_{00}^{*} Y_{11}^{*} Y_{11} \right|^2 = \frac{\langle r \rangle^2}{3}$$

$$\langle r \rangle = \int_0^\infty R_{21} R_{10} \, r \, (r^2 dr) = \sqrt{\frac{2}{3}} \frac{2^7}{3^4} a_0 = \text{radial integral}$$

from Lec. 22

$$\frac{dR_{+z}}{d\Omega} = \frac{|\langle r \rangle|^2}{3} \frac{1}{2\pi} \frac{\alpha}{c^2 \hbar^3} E_\gamma^3$$

For state $|2,1,1\rangle$ to emit photon in arbitrary direction $\vec{k}$, rotate state to align $\vec{k}$. Add factor $\frac{1+\cos\theta}{2}$

see p 3.19, p. 108

$$\frac{dR_+}{d\Omega_{\vec{k}}} = \frac{|RI|^2}{3} \frac{1}{2\pi} \frac{\alpha}{c^2\hbar^3} E_\gamma^3 \frac{(1+\cos\vartheta)^2}{4}$$

total rate includes probability to emit γ of either polarization:

$$\frac{dR}{d\Omega} = \frac{dR_+}{d\Omega_{\vec{k}}} + \frac{dR_-}{d\Omega_{\vec{k}}} = \frac{|RI|^2}{3}\left(\frac{1}{2\pi}\right)\frac{\alpha}{c^2\hbar^3} E_\gamma^3 \frac{(1+\cos^2\vartheta)}{2}$$

$$\int d\Omega \left(\frac{1+c^2}{2}\right) = \frac{8\pi}{3}$$

$$R = \frac{1}{3}\left(\frac{2}{3}\right)\frac{2^{14}}{3^8}\left(\frac{8\pi}{3}\right)\frac{1}{2\pi}\frac{\alpha a_0^2 E_\gamma^3}{c^2\hbar^3}$$

$$= \frac{2^{17}}{3^{11}} \frac{\alpha a_0^2}{c^2}\left(\frac{E_\gamma^3}{\hbar^3}\right)$$

with $E_\gamma = E_{2p} - E_{1s} = \frac{1}{2} mc^2\alpha^2 \left(1 - \frac{1}{2^2}\right) = \frac{3}{8} mc^2\alpha^2$

$$R_{2p\to 1s} = \left(\frac{2}{3}\right)^8 \alpha^5 \frac{mc^2}{\hbar}$$

$$\tau = \frac{1}{R} = 1.6 \text{ ns}$$