Phys 492, Spring 2010 M. Gold

Lecture 6: Nuclear Physics

Fundamental physics Quantum Chromodynamics (QCD) highly nonlinear theory that cannot be solved to yield potential. Approximate models (effective theories) based on meson (strongly interacting boson) exchange yield good description.

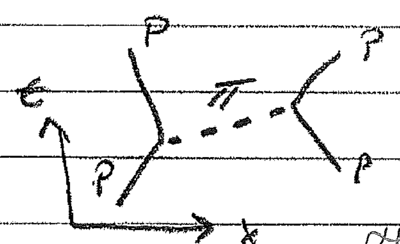
Basic properties of strong-nuclear force:

① short range

$$V(r) = \frac{g^2}{r} e^{-r(m_\pi c/\hbar)}$$

range set by π-meson (pion) Compton wavelength.

Virtual π exchange



Feynman space-time

$$\underbrace{(c\Delta t)}(m_\pi c^2) = \hbar c$$

range set by uncertainty principle.

$$\frac{\hbar c}{m_\pi c^2} = \frac{200 \text{ MeV}\cdot\text{fm}}{140 \text{ MeV}} \approx \text{fm}$$

② Strong coupling

$$\alpha_S = \frac{g^2}{\hbar c} \approx 15 \gg \alpha_{Em} = \frac{1}{137}$$

expect binding energies $\sim \alpha_S^2 = (2\times10^3)^2 \alpha_{Em}$

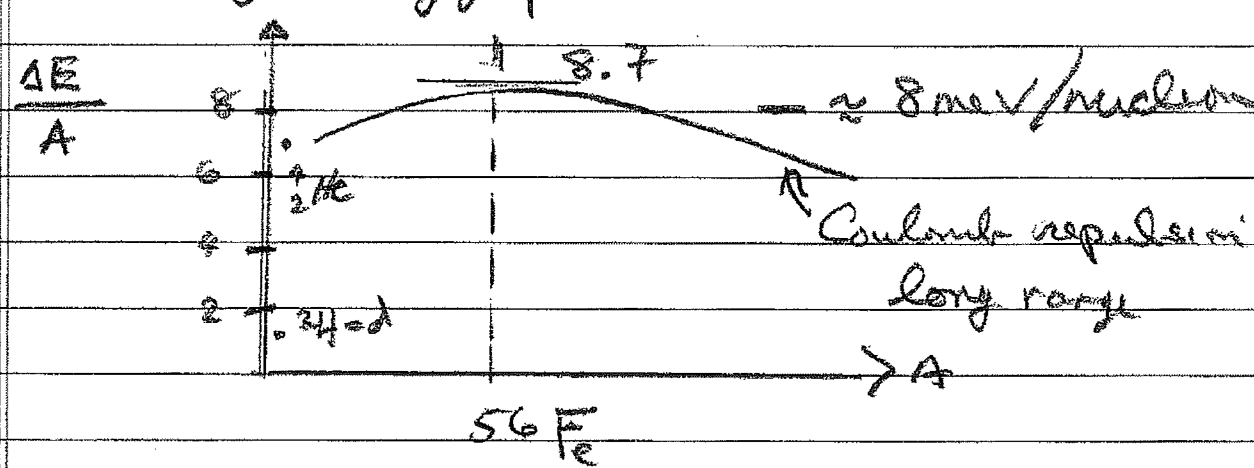
Order of magnitude $= 10^6 \times$ (10 eV atomic) $=$ 10 MeV,

③ nuclear radius

$r_A = (1.2\,\text{fm})\,A^{1/3}$ like dense sphere packing

nuclear density $\approx 10^{15}$ g/cm^3

④ Binding energy per nucleon

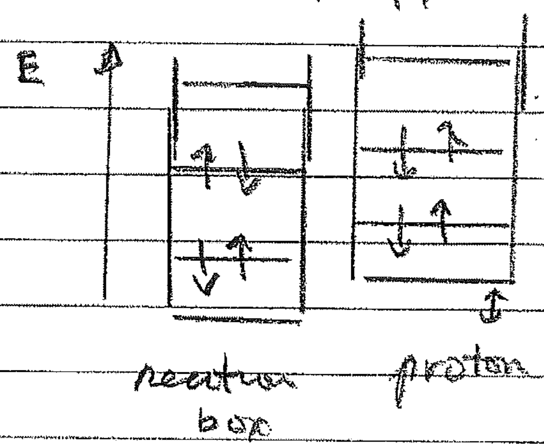


(i) saturation due to short range

(ii) long range Coulomb ⇒ more neutrons needed to bind heavy nuclei

⑤ Charge independence

light nuclei have $Z \approx (A-Z)$.
result of Pauli exclusion principle
and "charge independence" of nuclear
forces verified in
nn, np, pp scattering



Equal Z, N-Z minimizes energy

"charge independence" can be implemented
formally as iso-spin symmetry

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \qquad I = \tfrac{1}{2}$$

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \qquad I = 1$$

Write nuclear Hamiltonian in isospin
invariant form. States must be
completely antisymmetric wave function:

$$\psi = (\text{space})(\text{spin})(\text{isospin})$$

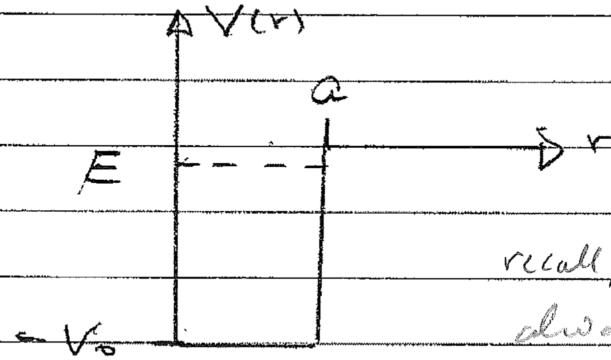
Finite Spherical Well model of deuteron

$d = np$ very weakly bound (few MeV)
only ground state ($\ell = 0$) exists

$$\psi_0 = \frac{u}{r}\frac{1}{\sqrt{4\pi}}$$

Finite spherical well potential model:

$$V(r) = \begin{cases} -V_0 & , r < a \\ 0 & , r > a \end{cases}$$



recall, in 1D bound state always exists. Not so in 3D

radial equation for $u(r)$ ($\ell = 0$)

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} - V(r)\,u(r) = E\,u(r)$$

define $q = \sqrt{2\mu|E|}/\hbar$ $\quad r > a$

$Q \equiv \sqrt{2\mu(V_0 - |E|)}/\hbar$ $\quad r < a$

general solution

$$U(r) = \begin{cases} A \sin Qr + \underset{\substack{0 \text{ to satisfy } U(0)=0 \\ \text{boundary}}}{B} \cos Qr & r<a \\ Ce^{-qr} + \underset{0 \text{ normalizable}}{D} e^{+qr} & r>a \end{cases}$$

just as for Coulomb case, $U \underset{r\to 0}{\longrightarrow} r^{\ell+1}$

Suppose $U(0) = \text{constant}$.

then $\psi \underset{r\to 0}{\longrightarrow} \dfrac{\text{const}}{r}$

But $\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})}$ which does

not solve finite well differential equation.

(or Coulomb potential for that matter)

proof: divergence theorem

$$\int_V d^3r\, \nabla^2\left(\frac{1}{r}\right) = \oint_S d^2A\, \hat{n}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

choose V as a sphere of radius r:

$$\oint d^2A\, \hat{r}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \int d^2A \frac{d}{dr}\left(\frac{1}{r}\right) = -\int \frac{d^2A}{r^2} = -4\pi$$

$$= -4\pi \int d^3r\, \delta^3(\vec{r})$$

transcendental equation

match boundary conditions

$$A \sin Qa = C e^{-q a}$$
$$A Q \cos Qa = -q C e^{-q a}$$

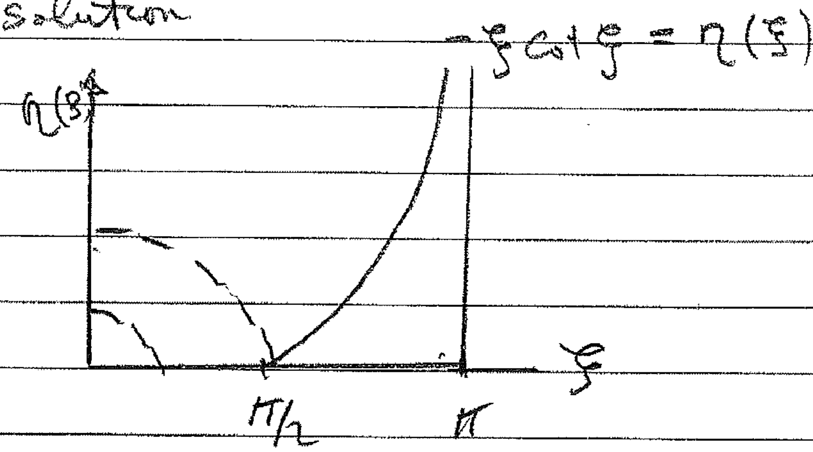
$$aQ \cot(aQ) = -q a$$

define $\eta \equiv q a$, $\xi \equiv Qa$

$$\xi \cot \xi = -\eta$$

$$\xi^2 + \eta^2 = \frac{2\mu}{\hbar^2} V_0 a^2 > \left(\frac{\pi}{2}\right)^2$$

graphical solution



__no__ bound state for $\xi^2 + \eta^2 < \left(\pi/2\right)^2$

in contrast to 1D square well

Estimate of V_0

photo-disintegration of d gives $E_b = 2.2$ MeV

$$\gamma + d \to n + p$$

So $$\frac{2\mu}{\hbar^2} V_0 a^2 \simeq \left(\frac{\pi}{2}\right)^2$$

exp. value of $a = 1.7$ fm

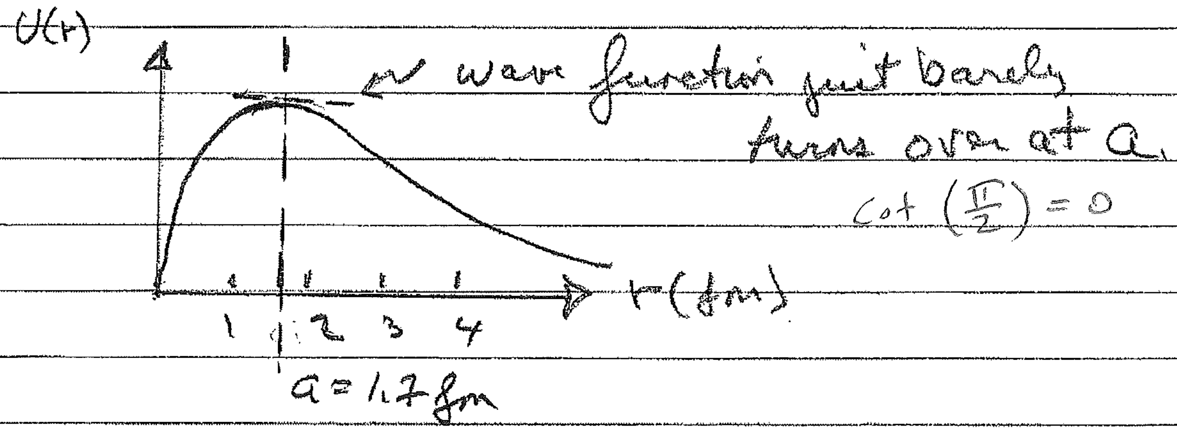
$$V_0 a^2 = \left(\frac{\pi}{2}\right)^2 \frac{(197 \text{ MeV fm})^2}{938 \text{ MeV}} \simeq 100 \text{ MeV} \cdot \text{fm}^2$$

$$V_0 = 35 \text{ MeV}$$

$$Q^{-1} = \frac{\hbar c}{\sqrt{2\mu c^2 (V_0 - |E|)}} = \frac{200 \text{ MeV} \cdot \text{fm}}{\sqrt{938 \text{ MeV} \cdot (35-2) \text{ MeV}}}$$

$$= 1.1 \text{ fm}$$

$$q^{-1} = \frac{\hbar c}{\sqrt{938 \text{ MeV} (2.2 \text{ MeV})}} = 4.4 \text{ fm}$$



more realistic nuclear force

Nuclear force is "charge-independent" isospin conserving, but spin dependent.

meson-theoretic two nucleon potential, one pion exchange in static limit.

spin, isospin operators:

σ_i - Pauli matrices acting on spin state

τ_i - Pauli matrices acting on isospin state

dimensionless separation variable $X = r \underbrace{\left(\frac{m_\pi c^2}{\hbar c}\right)}_{\sim 1/fm}$

spin tensor operator:

$$\hat{S}_{12} = 3(\vec{\sigma}_1 \cdot \hat{e}_r)(\vec{\sigma}_2 \cdot \hat{e}_r) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

then (See for example Roy, Nigam Nuclear Physics, Wiley)

$$V(x) = \underbrace{\frac{1}{3}\left(\frac{g^2}{4\pi}\right) m_\pi c^2}_{\text{constant } \equiv A} \vec{\tau}_1 \cdot \vec{\tau}_2 \left\{ \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \hat{S}_{12}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \right\} \frac{e^{-x}}{x}$$

Wave function of 2 nucleon system

$$\Psi_{NN} = \Psi_{space}\, \chi_{spin}\, \chi_{isospin}$$

In isospin formalism proton is isospin ↑, neutron isospin ↓

NN treated as identical particles.

Pauli-exclusion principle $\Psi_{NN}(1,2) = -\Psi_{NN}(2,1)$
completely anti-symmetric.

Expect ground state to be $l=0$ symmetric.

Aside: Actually deuteron has small $l=2$ component, giving it a small electric quadrupole moment, $Q \cong 0.3\ e\text{-}fm^2$

Isospin $I=0$ odd $\quad \chi_0^I = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ (deuteron d

$I=1$ even $\quad \chi_{1,1}^I = \uparrow\uparrow$

$\chi_{1,0}^I = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

$\chi_{1,-1}^I = \downarrow\downarrow$ di-neutron

deuteron wave function:

$$\psi_d \cong \frac{u(r)}{r} Y_{00}\, \chi_0^I \chi_{1,m}^S \qquad I=0,\ S=1$$

di-neutron wave function:

$$\psi_{nn} = \frac{u(r)}{r} Y_{00}\, \chi_{1,-1}^I \chi_0^S \qquad I=1,\ S=0$$

recall $2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2 \qquad \hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$

with $\hat{\vec{S}}_1 = \frac{\hbar}{2}\vec{\sigma}_1$

$$2\langle(\tfrac{1}{2}\vec{\sigma}_1 \cdot \tfrac{1}{2}\vec{\sigma}_2)\rangle = S(S+1) - 3/2$$

$$\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle = 2\left[S(S+1) - 3/2\right] = \begin{cases} -3 & S=0 \\ +1 & S=1 \end{cases}$$

similarly for $\langle \vec{\tau}_1 \cdot \vec{\tau}_2 \rangle$

tensor operator $\hat{S}_{12}$:

choose spin-quantization direction as $\hat{e}_r$

S = 0 singlet case

$$\langle \hat{S}_{12} \rangle = 3 \langle \underbrace{\vec{\sigma}_1 \cdot \hat{e}_r \, \vec{\sigma}_2 \cdot \hat{e}_r}_{-1} \rangle - \langle \underbrace{\vec{\sigma}_1 \cdot \vec{\sigma}_2}_{-3} \rangle = 0$$

S = 1 triplet case

$$\chi^S_{1,\pm 1} \qquad \langle \hat{S}_{12} \rangle = 3 \langle \underbrace{\vec{\sigma}_1 \cdot \hat{e}_r \, \vec{\sigma}_2 \cdot \hat{e}_r}_{+1} \rangle - \langle \underbrace{\vec{\sigma}_1 \cdot \vec{\sigma}_2}_{+1} \rangle = 2$$

$$\chi^S_{1,0} \qquad \langle \hat{S}_{12} \rangle = 3 \langle \underbrace{\vec{\sigma}_1 \cdot \hat{e}_r \, \vec{\sigma}_2 \cdot \hat{e}_r}_{-1} \rangle - \langle \underbrace{\vec{\sigma}_1 \cdot \vec{\sigma}_2}_{+1} \rangle = -4$$

then deuteron bound state $(\chi^S_{1,\pm 1})$ $I = 0, S = 1$

$$V_d(x) = A\,(-3) \left\{ 1 + 2\left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \right\} \frac{e^{-x}}{x}$$

$$= A(-3) \left\{ 3 + \frac{6}{x} + \frac{6}{x^2} \right\} \frac{e^{-x}}{x}$$

di-neutron state $I = 1, S = 0$

$$V_{nn}(x) = A(+1) \{ -3 \} \frac{e^{-x}}{x}$$

$\boxed{V_d \sim 3 V_{nn}}$ since d is weakly bound

$\gamma + d \to p + n$ threshold 2.2 MeV

$\Rightarrow$ di-neutron is not bound.