

Lecture #8 3D Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + \frac{1}{2} m\omega^2 \hat{r}^2 = \hat{H}_x + \hat{H}_y + \hat{H}_z$$

So solutions factorize in Cartesian coordinates:

$$\psi(\vec{r}) = X(x)Y(y)Z(z)$$

$$E = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega$$

To understand shell structure we need spherically symmetric solutions.

$$\psi(\vec{r}) = \frac{U(r)}{r} Y_{\ell m}(\theta, \phi)$$

Define dimensionless variables

$$\rho \equiv \sqrt{\frac{\mu\omega}{\hbar}} r, \quad \lambda \equiv \frac{2E}{\hbar\omega}$$

to get radial equation

$$U'' - \frac{\ell(\ell+1)}{\rho^2} U = -\lambda U$$

at small  $\rho$ ,  $U \sim \rho^{\ell+1}$  as  $\rho \rightarrow 0$

$R \rightarrow r^\ell$   
 $r=0$

as for Coulomb

at large  $\rho$

$$U'' = \rho^2 U$$

$$U \sim e^{-\rho^2/2}$$

as  $\rho \rightarrow \infty$

substitute  $U = \rho^{\ell+1} e^{-\rho^2/2} f(\rho)$

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$$u' = [(l+1)s^l - s^{l+2}] e^{-s^2/2} f + s^{l+1} e^{-s^2/2} f'$$

$$= s^{l+1} e^{-s^2/2} \left\{ \left( \frac{l+1}{s} - s \right) f + f' \right\}$$

$$u'' = s^{l+1} e^{-s^2/2} \left\{ \left( \frac{l+1}{s} - s \right)^2 f + 2f' \left( \frac{l+1}{s} - s \right) \right.$$

$$\left. + f'' + f \left( -\frac{(l+1)}{s^2} - 1 \right) \right\}$$

leading to

$$\boxed{(\lambda - 2l + 3)f + 2f' \left( \frac{l+1}{s} - s \right) + f'' = 0}$$

power series solution let  $f = \sum_{n=0}^{\infty} C_n s^n$

$$f' = \sum_{n=0}^{\infty} (n+1) C_{n+1} s^n$$

$$f'' = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} s^n$$

$$\left[ \left( \lambda - 2l - 3 \right) C_n s^n + 2(n+1) C_{n+1} s^n \left( \frac{l+1}{s} - s \right) \right.$$

$$\left. + (n+2)(n+1) C_{n+2} s^n \right] = 0$$

Terms with  $\frac{l+1}{s}$ :

$$\sum_{n'=-1}^{\infty} 2(n'+2)(l+1) s^{n'} C_{n'+2} = \sum_{n'=-1}^{\infty} 2(n'+2)(l+1) s^{n'} C_{n'+2}$$

$$n' = n - 1$$

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Continuing with  $\frac{l+1}{s}$  term

$$= 2 \frac{C_1}{s} (l+1) + \sum_{n=0}^{\infty} 2(n+2)(l+1) C_{n+2}$$

-  $g$  term:

$$- \sum_{n=0}^{\infty} 2(n+1) C_{n+1} g^{n+1} = - \sum_{n'=1}^{\infty} 2n' C_{n'} g^{n'}$$
$$- \sum_{n=0}^{\infty} 2n C_n g^n$$

then we have

$$\left. \frac{C_1}{s} (l+1) + \sum_{n=0}^{\infty} g^n \right\} (\lambda - 2l - 3) C_n - 2n C_n$$
$$\left. [2(n+2)(l+1) + (n+2)(n+1)] C_{n+2} \right\} = 0$$

Since  $g$  cannot be eliminated from  $C_1$  terms

$C_1 = 0$ . Then

$$\sum_{n=0}^{\infty} g^n \left\} (\lambda - 2n - 2l - 3) C_n + (n+2)(n+1+2l+2) C_{n+2} \right\} = 0$$

connects  $C_{n+2}$ ,  $C_n$ .  $C_1 = 0$  implies all odd coefficients are zero.

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recursion relation is

$$\frac{C_{n+2}}{C_n} = \frac{2n+2l+3-\lambda}{(n+2)(n+3+2l)} \xrightarrow{\text{large } n} \frac{2}{n}$$

at large  $n$ , this is the same as expansion of

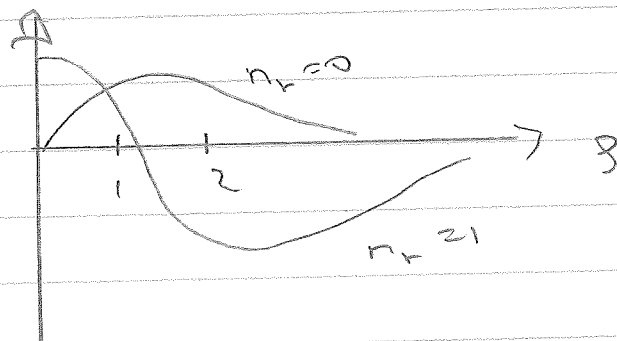
$$e^{-\rho^2} = \sum_{\text{even } n} \frac{2^{n/2} \rho^n}{(\frac{n}{2})!} \xrightarrow{\text{recursion}} \frac{2}{n+1}$$

Series must terminate and

$$\lambda = 2(n+l+\frac{3}{2}) = \frac{2E}{\hbar\omega} \quad n=0, 2, 4$$

or  $n=2n_r$ ,  $n_r=0, 1, 2$  (# radial nodes)

Sketch of radial wave functions



On the you find that  $s$  states with same  $l$  are degenerate. Just as for  $1/r$  potential  $r^2$  potential has classically closed orbits and quantum dynamical symmetry.

You find closed shells 2, 8, 20, 40 giving first 3 magic numbers.

To do better add spin-orbit term.

nuclei with closed shells have  $J=0$  ( $\frac{A}{2} N$  rotator)

$^{16}_8\text{O}$ ,  $^{40}_{20}\text{Ca}$ ,  $^{208}_{82}\text{Po}$

nuclei with one nucleon missing have  $J = \frac{1}{2}$

$^{15}_7\text{N}$ ,  $^{39}_{19}\text{K}$ ,  $^{207}_{82}\text{Pb}$

# from Eisberg and Resnic L-S coupling

