Physics 492 Spring 2010
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Lecture #8 3D Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{r}^2 = \hat{H}_x + \hat{H}_y + \hat{H}_z$$

So solutions factorize in cartesian coordinates:

$$\psi(\vec{r}) = X(x)Y(y)Z(z)$$

$$E = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

To understand shell structure we need spherically symmetric solutions.

$$\psi(\vec{r}) = \frac{u(r)}{r} Y_{\ell m}(\theta, \phi)$$

Define dimensionless variables

$$\rho \equiv \sqrt{\frac{\mu\omega}{\hbar}}\, r \quad , \quad \lambda \equiv \frac{2E}{\hbar\omega}$$

to get radial equation

$$u'' - \frac{\ell(\ell+1)u}{\rho^2} - \rho^2 u = -\lambda u$$

at small ρ, $u \underset{\rho \to 0}{\sim} \rho^{\ell+1}$ $\qquad R \underset{r \to 0}{\rightarrow} r^\ell$ as for Coulomb

at large ρ $\quad u'' = \rho^2 u$

$$u \underset{\rho\to\infty}{\sim} e^{-\rho^2/2}$$

substitute $\quad u = \rho^{\ell+1} e^{-\rho^2/2} f(\rho)$

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$$u' = \left[(\ell+1)\rho^{\ell} - \rho^{\ell+2}\right] e^{-\rho^2/2} f + \rho^{\ell+1} e^{-\rho^2/2} f'$$

$$= \rho^{\ell+1} e^{-\rho^2/2} \left\{ \left(\frac{\ell+1}{\rho} - \rho\right) f + f' \right\}$$

$$u'' = \rho^{\ell+1} e^{-\rho^2/2} \left\{ \left(\frac{\ell+1}{\rho} - \rho\right)^2 f + 2f' \left(\frac{\ell+1}{\rho} - \rho\right) \right.$$

$$\left. + f'' + f\left(-\frac{(\ell+1)}{\rho^2} - 1\right) \right\}$$

leading to

$$\boxed{(\lambda - 2\ell + 3) f + 2f'\left(\frac{\ell+1}{\rho} - \rho\right) + f'' = 0}$$

power series solution let $f = \sum_{n=0}^{\infty} c_n \rho^n$

$$f' = \sum_{n=0}^{\infty} (n+1) c_{n+1} \rho^n$$

$$f'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} \rho^n$$

$$\sum \left\{ (\lambda - 2\ell - 3) c_n \rho^n + 2(n+1) c_{n+1} \rho^n \left(\frac{\ell+1}{\rho} - \rho\right) \right.$$

$$\left. + (n+2)(n+1) c_{n+2} \rho^n \right\} = 0$$

Terms with $\frac{\ell+1}{\rho}$:

$$\sum 2(n+1)(\ell+1) c_{n+1} \rho^{n-1} = \sum_{n'=-1}^{\infty} 2(n'+2)(\ell+1) \rho^{n'} c_{n'+2}$$

$$n' = n-1$$

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Continuing with $\frac{\ell+1}{\rho}$ term

$$= \frac{2C_1}{\rho}(\ell+1) + \sum_{n=0}^{\infty} 2(n+2)(\ell+1)\, C_{n+2}\, \rho^n$$

- ρ term:

$$-\sum_{n=0}^{\infty} 2(n+1)\, C_{n+1}\, \rho^{n+1} = -\sum_{n'=1}^{\infty} 2n'\, C_{n'}\, \rho^{n'} \qquad n' = n+1$$

$$-\sum_{n=0}^{\infty} 2n\, C_n\, \rho^n$$

then we have

$$\frac{C_1}{\rho}(\ell+1) + \sum_{n=0}^{\infty} \rho^n \Big\{ (\lambda - 2\ell - 3) C_n - 2n\, C_n$$
$$\big[2(n+2)(\ell+1) + (n+2)(n+1)\big]\, C_{n+2} \Big\} = 0$$

Since ρ cannot be eliminated from C_1 terms, $\underline{C_1 = 0}$. Then

$$\sum_{n=0}^{\infty} \rho^n \Big\{ (\lambda - 2n - 2\ell - 3) C_n + (n+2)(n+1+2\ell+2) C_{n+2} \Big\} = 0$$

connecting C_{n+2}, C_n. $C_1 = 0$ implies all odd coefficients are zero.

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recursion relation is

$$\frac{c_{n+2}}{c_n} = \frac{2n+2\ell+3-\lambda}{(n+2)(n+3+2\ell)} \xrightarrow[\text{large } n]{} \frac{2}{n}$$

at large n, this is the same as expansion of

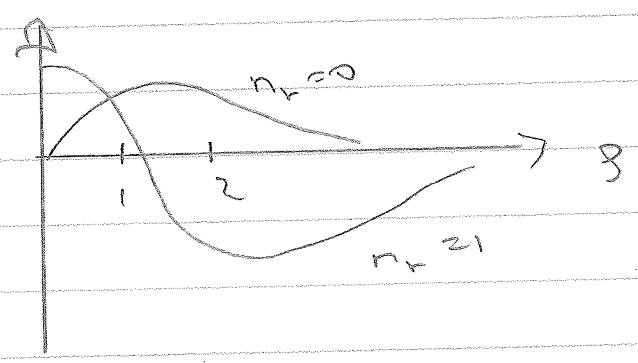
$$e^{\xi^2} = \sum_{n\,\text{even}} \frac{2^{n/2}\,\xi^n}{\left(\frac{n}{2}\right)!} \xrightarrow[\text{recursion}]{} \frac{2}{n+1}$$

Series must terminate and

$$\lambda = 2\left(n+\ell+3/2\right) = \frac{2E}{\hbar\omega} \qquad n = 0, 2, 4$$

or $n = 2n_r$, $n_r = 0, 1, 2$ (# radial nodes)

Sketch of radial wave function



On this you find that states with same l are degenerate. Just as for $1/r$ potential r^2 potential has classically closed orbits and quantum dynamical symmetry.

You find closed shells 2, 8, 20, 40 giving first 3 magic numbers.

To do better add spin-orbit term.

nuclei with closed shells have $J=0$ ($^A_Z N$ notation)

$^{16}_{8}O$, $^{40}_{20}Ca$, $^{208}_{82}Pb$

nuclei with one nucleon missing have $J=\frac{1}{2}$

$^{15}_{7}N$, $^{39}_{19}K$, $^{207}_{82}Pb$

from Eisberg and Resnic L-S coupling