Spring 2016

Physics 492

Recitation #3 Solutions

(1) $a_0 = \frac{\hbar}{\alpha \mu c}$; $a_0^{e^+e^-} = 2a_0$

(2) $|\langle L_z \rangle| = m\hbar$

$$L_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \quad L_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$$

$$\langle L_x \rangle = 0 = \langle L_y \rangle$$

$$\langle \hat{L}_x^2 \rangle = \langle \left| \tfrac{1}{4}(\hat{L}_+^2 + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+ + \hat{L}_-^2) \right\rangle$$

$$= \frac{1}{4}\langle \hat{L}_+\hat{L}_- \rangle + \frac{1}{4}\langle L_- L_+ \rangle$$

$$\hat{L}_+\hat{L}_- |l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\,\hat{L}_+|l,m-1\rangle$$
$$= \hbar^2\big(l(l+1) - m(m-1)\big)|l,m\rangle$$

$$\hat{L}_-\hat{L}_+ |l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,\hat{L}_-|l,m+1\rangle$$
$$= \hbar^2\big(l(l+1) - m(m+1)\big)|l,m\rangle$$

$$\langle \hat{L}_x^2 \rangle = \frac{\hbar^2}{4}\left[2l(l+1) - 2m^2\right] = \frac{\hbar^2}{2}\left(l(l+1) - m^2\right)$$

$$\langle \hat{L}_y^2 \rangle = +\frac{1}{4}\langle \hat{L}_-\hat{L}_+ \rangle + \frac{1}{4}\langle \hat{L}_+\hat{L}_- \rangle = \langle \hat{L}_x^2 \rangle$$

$$\Delta L_x \Delta L_y = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right] \leq \frac{l\hbar^2}{2}$$

so $\frac{l\hbar^2}{2} \geq \frac{m\hbar^2}{2}$ ✓

(3) <u>Ehrenfest's theorem</u>

$$\frac{d}{dt}\langle \vec{L} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \vec{L}] \rangle$$

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + V \qquad \vec{L} = \vec{r} \times \vec{p}$$

identity $[\hat{a}+\hat{b}, \hat{c}] = [\hat{a}, \hat{c}] + [\hat{b}, \hat{c}]$

$$[\hat{H}, \vec{L}] = \frac{1}{2\mu}[\hat{p}^2, \vec{L}] + [V, \vec{L}]$$

$$[\hat{p}^2, L_i] = [\hat{p}^2, \sum_{j,k} \varepsilon_{ijk} X_j P_k]$$

$$= \sum_{j,k} \varepsilon_{ijk} [\hat{p}^2, X_j P_k]$$

$$= \sum_{jk} \varepsilon_{ijk} \left(\hat{p}^2 X_j P_k - X_j P_k P^2\right)$$

since $[\hat{p}^2, P_k] = 0$, $= \sum_{jk} \varepsilon_{ijk} [\hat{p}^2, X_j] \hat{P}_k$

we proved $[f(\hat{p}_j), X_j] = -i\hbar\left(\frac{\partial f}{\partial P_j}\right)$

so $[\hat{p}^2, X_j] = -2i\hbar P_j$.

$$[\hat{p}^2, L_i] = -2i\hbar \sum_{jk} \varepsilon_{ijk} P_j P_k = 0$$

second commutator

$$[V, \hat{L}_i] = [V, (\vec{r}\times\vec{p})_i] = \sum_{jk} \varepsilon_{ijk} [V, x_j p_k]$$

$$= \sum_{jk} \varepsilon_{ijk} x_j [V, \frac{\hbar}{i}\nabla_k]$$

$$= -\frac{\hbar}{i} \sum_{jk} \varepsilon_{ijk} x_j \nabla_k V = -\frac{\hbar}{i}(\vec{r}\times\vec{\nabla}V)_i$$

finally

$$\frac{d}{dt}\langle \vec{L} \rangle = \langle \vec{r}\times(-\vec{\nabla}V)\rangle = \langle \vec{N} \rangle$$

(4) $$\hat{H} = \frac{1}{2I_1}(\hat{L}_x^2 + \hat{L}_y^2) + \frac{1}{2I_3}\hat{L}_z^2$$

where z-direction is symmetry axis of NH_3

$$\hat{H} = \frac{1}{2I_1}(\hat{L}^2 - \hat{L}_z^2) + \frac{1}{2I_3}\hat{L}_z^2$$

so $[\hat{H}, \hat{L}_z] = 0$, $[\hat{H}, \hat{L}^2] = 0$

$$\hat{H}|l, m\rangle = \frac{\hbar^2}{2}\left[\frac{l(l+1) - m^2}{I_1} + \frac{m^2}{I_3}\right]|l, m\rangle$$