Tunneling in Path Integral (Shankar 21.2)

p613

Propagator

$$U(x,x',t) = \int \mathcal{D}' \exp\left[\frac{i}{\hbar}\int dt\left(\frac{m}{2}\frac{dx^2}{dt^2} - V\right)\right]$$

$t \to -i\tau$ complex time τ $\quad \dot{x} \equiv \frac{dx}{d\tau}$

$$U(x,x',\tau) = \int \mathcal{D}'' \exp\left[\frac{1}{\hbar}\int d\tau\left(-\frac{m}{2}\ddot{x} - V\right)\right]$$

See Shankar

$$= \int \mathcal{D}'' \exp\left[-\frac{1}{\hbar}\int d\tau\left(\frac{m}{2}\ddot{x} + V\right)\right]$$

Euclidean $\mathcal{L}_E = \frac{m}{2}\ddot{x} + V$ $\qquad$ V changes sign

note [name comes from Minkowski Euclidean time in S.R. $\quad (\Delta s)^2 = \Delta x^2 - (c\Delta t)^2 = (\Delta x)^2 + (c\Delta\tau)^2$
$\Delta t = ic\Delta\tau$]

free particle propagator,

$$U(x,x;t) = \left[\frac{m}{2\pi\hbar i t}\right]^{1/2} \exp\left[\frac{im(x-x')^2}{2\hbar t}\right]$$

$$\xrightarrow[t=-i\tau]{} \left[\frac{m}{2\pi\hbar\tau}\right]^{1/2} \exp\left(\frac{-m(x-x')^2}{2\hbar\tau}\right)$$

time evolution operator

$$U(t) = U\left(-\frac{iHt}{\hbar}\right) \underset{t=-i\tau}{\longrightarrow} U(\tau) = \exp\left(-\frac{H\tau}{\hbar}\right)$$

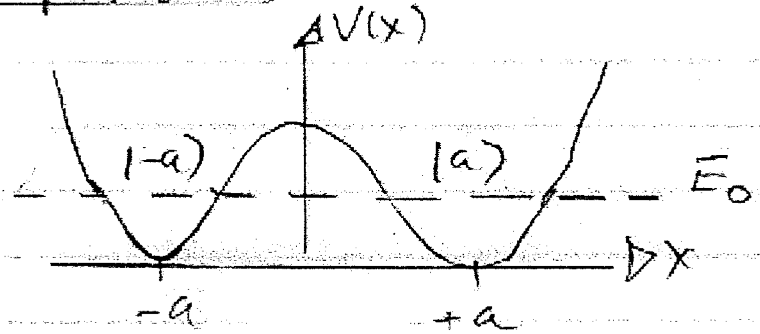
non-unitary but Hermitian. Norm of state not preserved. Every state evolves to ground state. Expand in energy eigenstates

$$\langle x|U|x\rangle = \sum_n \langle x|n\rangle\langle n|U|x\rangle$$

$$= \sum_n \langle x|n\rangle\langle n|x\rangle \exp\left(-\frac{E_n\tau}{\hbar}\right)$$

$$\underset{\lim \tau\to\infty}{\longrightarrow} \quad \langle x|0\rangle\langle 0|x\rangle \exp\left(-\frac{E_0\tau}{\hbar}\right)$$

<u>double well potential</u>



$$V(x) = A^2(x^2 - a^2)^2$$

ground state $|\pm a\rangle$

$\langle +a | -a \rangle \cong 0$

$$[H] = \begin{bmatrix} \langle a|H|a\rangle & \langle a|H|-a\rangle \\ \langle -a|H|a\rangle & \langle a|H|-a\rangle \end{bmatrix}$$

$$E_0 = \langle a|H|a\rangle$$

extract off diagonal tunneling as

$$\langle a|U(\tau)|-a\rangle = \langle a|\exp\left(\frac{-H\tau}{\hbar}\right)|-a\rangle$$

$$\simeq -\frac{\tau}{\hbar}\langle a|H|-a\rangle$$

propagator:

we can ignore

$$\langle a|U(\tau)|-a\rangle = A\exp\left(-\frac{1}{\hbar}\int_0^\tau d\tau\, \mathcal{L}_E\right)$$

Classical path conserves Euclidean energy

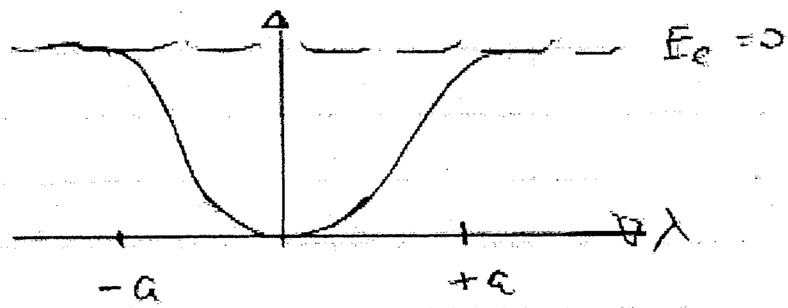
$$E_e = \frac{m\dot{x}^2}{2} - V$$

$$\dot{x} = \frac{dx}{d\tau} = \sqrt{(E_e + V)\frac{2}{m}}$$

$$\tau(x) = \int_0^x \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E_e + V}} + \tau_0$$

integration constant $x(\tau_0) = 0$

Take value $E_e = 0$



Classically takes ∞ time to get from $-a$ to a

let $y = x/a$

$$\tau - \tau_0 = \sqrt{\frac{m}{2}} \frac{1}{A} \frac{1}{a} \int \frac{dy}{\sqrt{y^2 - 1}} = \sqrt{\frac{m}{2}} \frac{1}{A} \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$x(\tau) = a \tanh\left[\sqrt{\frac{2}{m}} A a (\tau - \tau_0)\right]$$

tunneling occurs in instant (compared to ∞) at $\tau = \tau_0$ when $x = 0$.

Termed "instanton" solution by 't Hooft.

Summing over all paths for $-\frac{T}{2} < \tau_0 < \frac{T}{2}$ gives factor τ,

$$\langle a | U(\tau) | -a \rangle = \tau \exp\left(-\frac{1}{\hbar}\int_0^{\tau} \mathcal{L}_E \, d\tau'\right)$$

don't worry about units!

For $E_e = 0$, $\quad 0 = E_e = \frac{m\dot{x}^2}{2} - V$

$$\mathcal{L}_E = \frac{m\dot{x}^2}{2} + V = 2V = m\dot{x}$$

$$S_{cl} = \underset{\text{Classical path}}{\int \mathcal{L}_E \, d\tau} = \int_{-a}^{a} m\left(\frac{dx}{d\tau}\right)^2 \frac{dx}{\frac{dx}{d\tau}} = \int_{-a}^{a} m\left(\frac{dx}{d\tau}\right) dx$$

$$= \int_{-a}^{a} \sqrt{2m\,V(x)}\, dx \qquad S_{cl} \text{ semiclassical approximation}$$

We will see this is WKB tunneling result with $E = 0$

Now compare,

$$\langle a | U(\tau) | -a \rangle = \tau \exp\left(-\frac{1}{\hbar}\int_0^{\tau} \mathcal{L}_E \, d\tau\right) = \tau \exp\left(-\frac{S_{cl}}{\hbar}\right)$$

$$= -\tau \langle a | H | -a \rangle$$

off diagonal term is negative and equal to

$$\langle a | H | -a \rangle = -\hbar K \exp\left(-\frac{1}{\hbar} S_{cl}\right)$$

K constant with dim of energy