Lecture #19 - Coherent States

We found that $\langle x(t)\rangle$ for the simple harmonic oscillator Hamiltonian satisfies the classical equation of motion

$$\langle x(t)\rangle = \langle x(0)\rangle \cos(\omega t) + \frac{\langle p(0)\rangle}{m\omega} \sin(\omega t)$$

However, energy eigenstates have

$$\langle n|\hat{x}|n\rangle = 0 \quad \langle n|\hat{p}|n\rangle = 0$$

We can find superposition states that have non-zero $\langle \hat{x}\rangle$, $\langle \hat{p}\rangle$ and have minimum uncertainty product

$$\Delta x \, \Delta p = \frac{\hbar}{2}.$$

These states are called coherent states

(Schrödinger 1926; Roy Glauber, Physical Review 131 2766 1963)

Consider time evolution of $\langle a\rangle$:

$$\frac{d}{dt}\langle a\rangle = \frac{i}{\hbar}\langle [H, a]\rangle = \frac{i\hbar\omega}{\hbar}[a^+a, a] = -i\omega\langle \hat{a}\rangle$$

define $a(0) = \hat{a} \qquad a(t) = e^{-i\omega t} a$

then $\frac{d}{dt}\langle a(t)\rangle = -i\omega\langle a\rangle$

similarly $\quad a^+(t) = e^{+i\omega t} a^+(0)$

Simple time evolution suggests we consider eigenstates of $\hat{a}$:

$$\hat{a}|z\rangle = z|z\rangle \qquad \langle z|\hat{a}^{+} = \langle z|z^{*}$$

where z is complex because $\hat{a}$ is not hermitian.

Expand in energy eigenstate basis

$$|z\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|z\rangle$$

We find a recursion relation

$$a|z\rangle = z|z\rangle = \sum_{n=0}^{\infty} \sqrt{n}\,|n-1\rangle\langle n|z\rangle$$

$$= \sum_{n=0}^{\infty} \sqrt{n+1}\,|n\rangle\langle n+1|z\rangle$$

with orthogonality $\langle n|n'\rangle = \delta_{nn'}$

$$z\langle n|z\rangle = \sqrt{n+1}\,\langle n+1|z\rangle$$

or $$\langle n+1|z\rangle = \frac{z}{\sqrt{n+1}}\langle n|z\rangle$$

and thus

$$\boxed{\langle n|z\rangle = \frac{z^{n}}{\sqrt{n!}}\langle 0|z\rangle}$$

Also, substituting back into the expansion of $|z\rangle$ in terms of energy eigenstates

$$|z\rangle = \sum_{n=0}^{\infty} |n\rangle \frac{z^{n}}{\sqrt{n!}}\langle 0|z\rangle$$

$$\boxed{|z\rangle = \langle 0|z\rangle \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{n!}}|n\rangle}$$

Normalization

$$\langle z|z\rangle = 1 = |\langle 0|z\rangle|^2 \cdot \sum_{m,n=0}^{\infty} \frac{(z^*)^m z^n}{\sqrt{n!\,m!}} \langle m|n\rangle$$

$$= |\langle 0|z\rangle|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}$$

$$= |\langle 0|z\rangle|^2 e^{|z|^2}$$

giving

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

then since $|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$

$$|z\rangle = e^{-|z|^2/2} \sum \frac{(za^+)^n}{n!} |0\rangle$$

$$\boxed{|z\rangle = e^{-|z|^2/2} e^{za^+} |0\rangle}$$

note: These states are normalized but not orthogonal. They form an overcomplete basis.

From H.W. worksheet

$$P(n) \equiv |\langle n|z\rangle|^2 = e^{-|z|^2} \frac{|z|^{2n}}{n!}$$

a Poisson with mean $|z|^2$

$\sum P(n) = 1$

Similar to usual trick for Poisson,

$$\frac{d}{d|z|^2}\left(1 = \sum P(n)\right) = 0 = \sum_n \left(-P(n) + \frac{n}{|z|^2} P(n)\right)$$

mean $$\sum_{n=0}^{\infty} n\, P(n) = |z|^2 \sum_n P(n) = |z|^2$$

and $$\langle z|\hat{n}|z\rangle = \sum n\, P(n) = |z|^2$$

and since Variance = mean for Poisson,

$$(\Delta n)^2\Big|_{\text{state } |z\rangle} = |z|^2 = \bar{n} \Rightarrow \Delta n = \sqrt{\bar{n}}$$

uncertainty product

$$\Delta x \Delta p = \hbar\, \Delta y \Delta q$$

$$\hat{y} = \frac{1}{\sqrt{2}}(a^+ + a) \qquad \hat{q} = \frac{i}{\sqrt{2}}(a^+ - a)$$

$$\langle z|a|z\rangle = z \qquad \text{and}$$

$$(a|z\rangle)^+ = \langle z|a^+ = \langle z|z^*$$

$$\langle z|a^+|z\rangle = z^*$$

define $$z = \frac{1}{\sqrt{2}}(y_0 + i q_0)$$

where y_0, q_0 are real

Then for $|\psi(0)\rangle = |z\rangle$

$$\langle \hat{y} \rangle = \frac{1}{\sqrt{2}} \langle (z^* + z) \rangle = y_0$$
$$\langle \hat{q} \rangle = \frac{1}{\sqrt{2}} \langle (z^* - z) \rangle = q_0$$

$$\langle \hat{y}^2 \rangle = \langle z | \tfrac{1}{2}(a^\dagger + a)^2 | z \rangle = \tfrac{1}{2} \langle z | \big(a^{\dagger 2} + a^2 + \underbrace{a^\dagger a + a a^\dagger}_{2a^\dagger a + 1}\big) | z \rangle$$

$$= \tfrac{1}{2}(z^{*2} + z^2 - 2z^* z + 1) = \tfrac{1}{2}(z^* + z)^2 + \tfrac{1}{2} = y_0^2 + \tfrac{1}{2}$$

$$(\Delta y)^2 = \langle \hat{y}^2 \rangle - \langle \hat{y} \rangle^2 = \tfrac{1}{2}$$

similarly $(\Delta q)^2 = \frac{1}{2}$

$$\Delta x \Delta p = \hbar \Delta y \Delta q = \frac{\hbar}{2}$$

Suggests coherent states are Gaussians.

<u>Wave function:</u>

$$\phi_z(y) = \langle y | z \rangle$$

$$a|z\rangle = z|z\rangle \qquad a = \frac{1}{\sqrt{2}}(\hat{y} + i\hat{q})$$

$$\langle y | a | z \rangle = z \langle y | z \rangle$$

$$\frac{1}{\sqrt{2}} \langle y | y + i\left(\frac{1}{i}\frac{\partial}{\partial y}\right) | z \rangle = z \langle y | z \rangle$$

$$\frac{1}{\sqrt{2}}\left(y\phi_z + \frac{\partial}{\partial y}\phi_z\right) = z\,\phi_z$$

$$\uparrow \ \frac{1}{\sqrt{2}}(y_0 + iq_0)$$

$$\frac{\partial \phi_z}{\partial y} = -(y - y_0 - iq_0)\phi_z$$

$$\phi_z(y) = \text{const} \times \exp\left\{-(y - y_0 - iq_0)^2/2\right\}$$

$$(y-y_0-i\wp_0)^2/2 = \frac{1}{2}\left[(y-y_0)^2 - 2i\wp_0(y-y_0) + \wp_0^2\right]$$

goes in normalization

$$\phi_z(y) = e^{i\wp_0 y_0} \frac{1}{\pi^{1/4}} e^{i\wp_0 y} e^{-(y-y_0)^2/2}$$

ignore overall phase

Gaussian with $\langle \hat{y}(0)\rangle = y_0 \quad \langle \hat{\wp}(0)\rangle = \wp_0$

$$\boxed{\langle y|z\rangle = \frac{1}{\pi^{1/4}} e^{i\wp_0 y} e^{-(y-y_0)^2/2}}$$

These coherent states are also translations in position, momentum of the ground state

$$\hat{T}(y_0) = e^{-iy_0\hat{\wp}} \qquad \hat{W}(\wp_0) = e^{i\wp_0\hat{y}}$$

$$\langle y|\, W(\wp_0)T(y_0)|0\rangle = e^{i\wp_0 y}\langle y|T(y_0)|0\rangle$$

$$= e^{i\wp_0 y}\langle y-y_0|0\rangle$$

$$= \frac{e^{i\wp_0 y}}{\pi^{1/4}} e^{-(y-y_0)^2/2} \qquad = \langle y|z\rangle$$

You can also show that the translation operator can be written (up to irrelevent phase)

$$\exp(z a^{+} - z^{*} a)$$

<u>Time Dependence</u>

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

$$\hat{U}(t)|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} |n\rangle$$

$$= e^{-i\omega t/2} e^{|z|^2/2} \sum \frac{(z e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t/2} |z(t)\rangle$$

where $z(t) \equiv z e^{-i\omega t}$

Expand in terms of y_0, g_0:

$$z(t) = \frac{1}{\sqrt{2}}(y_0 + i g_0)(\cos\omega t - i \sin\omega t)$$

$$= \frac{1}{\sqrt{2}}(y_0 \cos\omega t + g_0 \sin\omega t)$$

Then $\langle y(t)\rangle = \langle z(t)|\hat{y}|z(t)\rangle$

$$= \frac{1}{\sqrt{2}}(z^{*}(t) + z(t))$$

$$\boxed{\langle y(z)\rangle = y_0 \cos\omega t + q_0 \sin\omega t}$$

coherently oscillating Gaussian

Classical Limit

take $m = 10\,g$ $\quad l = 100\,cm$ $\quad g = 10\,m/s^2$

$X_0 = 10\,cm$ $\quad P_0 = 0$

period $\tau = 2\pi\sqrt{\frac{l}{g}} = 0.63\,s$

$\omega = \frac{2\pi}{\tau} = 10\,rad/s$

$|z| = \sqrt{\frac{m\omega}{2\hbar}}\,X_0$ $\qquad \hbar = 10^{-34}\,J\cdot s$

$= \sqrt{5}\times 10^{16} \gg 1$

$\Delta p = \sqrt{\frac{m\hbar\omega}{2}} = 2\times 10^{-18}\,kg\,m/s$

$\Delta v = 2\times 10^{-16}\,m/s$

$\frac{\Delta v}{v_{max}} = \frac{2\times 10^{-16}\,m/s}{0.1\,m/s} = 2\times 10^{-15}$

$v_{max} = \dot{X}(t)\Big|_{t=\frac{n\omega}{2\pi}} = \omega X_0 = 0.1\,m/s$

$\frac{\Delta E}{\langle E\rangle} = \frac{1}{|z|} = 0.4\times 10^{-16}$ $\quad$ where $\langle E\rangle = 0.5\times 10^{-4}\,J$

Remark. A squeezed state in quantum optics has $\Delta x < \frac{\hbar}{2}$, $\Delta p > \frac{\hbar}{2}$ such that $\Delta x \Delta p = \frac{\hbar}{2}$.

Quantum optics (see for example R. Loudon, The Quantum Theory of Light, Clarendon press, 1979)

Quantum E.M. field is a collection of harmonic oscillators, one for each frequency mode ω and polarization $\vec{\varepsilon}$. Single photon state,

$$|1, \vec{k}, \vec{\varepsilon}\rangle$$

They are bosons, so we can put n of them in the same state. Here we describe an EM wave of momentum $\hbar\vec{k}$ and polarization $\vec{\varepsilon}$, so we drop these labels and write n-photon state as simply $|n\rangle$.

A classical EM wave has a definite phase.

The coherent state corresponds to the EM wave in the limit $|z| \gg 1$.

Phase Operator:

$z = |z| e^{i\phi}$ introduce hermitian operator $\hat{\phi}$

$$\hat{a} = \sqrt{\hat{n}+1}\, e^{i\hat{\phi}}$$

$$\hat{a}^{+} = e^{-i\hat{\phi}} \sqrt{\hat{n}+1}$$

$$\hat{a}\hat{a}^{+} = \sqrt{N+1}\, \underbrace{e^{i\hat{\phi}} e^{-i\hat{\phi}}}_{\hat{I}} = \hat{n}+1$$

Then $e^{i\hat{\phi}} = \cos\hat{\phi} + i\sin\hat{\phi} = (\hat{n}+1)^{-1/2}\hat{a}$

$e^{-i\hat{\phi}} = \cos\hat{\phi} - i\sin\hat{\phi} = (\hat{n}+1)^{-1/2} a^{+}$

Only non-zero expectation values are

$\langle n-1 | e^{i\hat{\phi}} | n \rangle = 1$

$\langle n+1 | e^{-i\phi} | n \rangle = 1$

<u>Uncertainty</u>

$[\hat{n}, e^{i\phi}] = [\hat{n}, (\hat{n}+1)^{-1/2}\hat{a}] = (\hat{n}+1)^{-1/2}(-\hat{a}) = -e^{-i\phi}$

and $[\hat{n}, e^{-i\phi}] = e^{i\hat{\phi}}$

from which we get

$[\hat{n}, \cos\hat{\phi}] = -i\sin\hat{\phi}$

$[\hat{n}, \sin\phi] = +i\cos\hat{\phi}$

from $\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2} |\langle \hat{A}, \hat{B} \rangle|$

$\Delta n\, \Delta\cos\phi \geq \frac{1}{2} |\langle \sin\phi \rangle|$

$\Delta n\, \Delta\sin\phi \geq \frac{1}{2} |\langle \cos\phi \rangle|$

define $(\Delta\phi)^2 = (\Delta\cos\phi)^2 + (\Delta\sin\phi)^2$

$$\boxed{\Delta n\, \Delta\phi \geq \frac{1}{2}}$$

Energy (photon number eigenstate $|n\rangle$, $\Delta n = 0$, $\Delta\phi \to \infty$

Coherent states:

$$|\langle n|z\rangle|^2 = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!} \quad \text{Poisson}$$

$$\langle \hat{n} \rangle = |z|^2$$

$$(\Delta n)^2 = |z|^2$$

In the $|z| >> 1$ limit

$$\Delta\phi = \frac{1}{2}\frac{1}{|z|}$$

$$\Delta n \, \Delta\phi = \frac{1}{2}$$

Classical EM wave corresponds to $|z|^2 = \langle n\rangle >> 1$

Coherent state,

Phase diagram