Phy 521 Fall 2023

Discrete Symmetries

① Parity $\hat{\Pi}$ spatial inversion

classical $\vec{r} \to -\vec{r}$

$$\hat{\Pi}^{+}\hat{x}\hat{\Pi} = -\hat{x}$$

then $\hat{x}\hat{\Pi}|x\rangle = -x(\hat{\Pi}|x\rangle)$

so $\hat{\Pi}|x\rangle = e^{i\delta}|-x\rangle$ we can choose δ by convention $= 0$

$$\Pi^{2}|x\rangle = |x\rangle \quad \text{so} \quad \Pi^{-1} = \Pi^{+} = \Pi$$

$\hat{\Pi}$ is hermitian.

Parity of momentum $\hat{P}_x$. Translation operator

$$T_x(a)|x\rangle = |x+a\rangle$$

$$T_x(-a)\,\hat{\Pi}|x\rangle = T_x(-a)|-x\rangle = |-x-a\rangle = \hat{\Pi}|x+a\rangle$$
$$= \hat{\Pi}\,\hat{T}(a)|x\rangle$$

or $\hat{T}_x(-a)\hat{\Pi} = \hat{\Pi}\hat{T}(a)$

infinitesimally,

$$\left(1 + \frac{i\varepsilon\hat{P}_x}{\hbar}\right)\hat{\Pi} = \hat{\Pi}\left(1 - \frac{i\varepsilon P_x}{\hbar}\right)$$

$$\hat{P}_x\hat{\Pi} = -\hat{\Pi}P_x$$

$$\Pi^{\dagger} P_x \Pi = -P_x$$

in verts operator $\vec{\hat{P}} \xrightarrow[\Pi]{} -\vec{\hat{P}}$

orbital $\vec{L} = \vec{r} \times \vec{p} \xrightarrow[\Pi]{} \vec{L}$

or $\Pi \vec{L} \Pi = \vec{L}$

Similarly for spin, $\Pi \vec{S} \Pi = S$

If $[\hat{\Pi}, \hat{H}] = 0$ energy eigenstate are states of definite parity.

$$\hat{\Pi} |\psi_E\rangle = \pm |\psi_E\rangle$$

for $|\psi_E\rangle = |n \ell m\rangle$

parity is property of spherical harmonics!

$\vec{r} \to -\vec{r}$ $\theta \to \pi - \theta$, $\phi \to \phi + \pi$

$$Y_\ell^m(\theta \phi) \to (-1)^\ell Y_\ell^m(\theta, \phi)$$

$$\hat{\Pi} |n \ell m\rangle = (-1)^\ell |n \ell m\rangle$$

5

Time reversal τ

classically $\vec{r} \xrightarrow[\tau]{} \vec{r}$, $\vec{v} \xrightarrow[\tau]{} -\vec{v}$, $\vec{p} \xrightarrow[\tau]{} -\vec{p}$

electric, magnetic fields:

charge $q \to q$ current $\vec{j} \to -\vec{j}$

$$\vec{E} \to \vec{E} \quad , \quad \vec{B} \to -\vec{B}$$

Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi$$

if $\psi(\vec{x},t)$ is solution, $\psi(\vec{x},-t)$ is not due to $\frac{\partial}{\partial t}$. However, $\psi^*(\vec{x},-t)$ is.
So τ includes <u>complex conjugation</u>.

$$\psi_E(\vec{x},t) = \psi_E(\vec{x},0)\, e^{-iE_n t/\hbar}$$

$$\xrightarrow[\tau]{} \psi_E^*(\vec{x},0)\, e^{-iE_n t/\hbar}$$

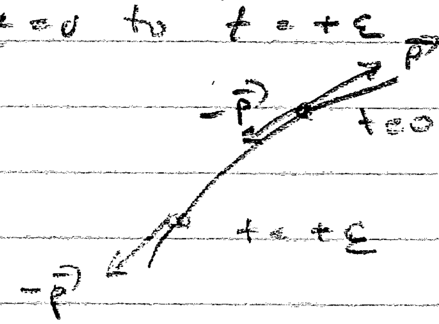
Acting on arbitrary ket, introduce operator

$$|\alpha\rangle \xrightarrow[\tau]{} \hat{\tau}|\alpha\rangle \qquad \hat{\tau} \text{ operator}$$

Consider infinitesimal time evolution of state (Sakurai)

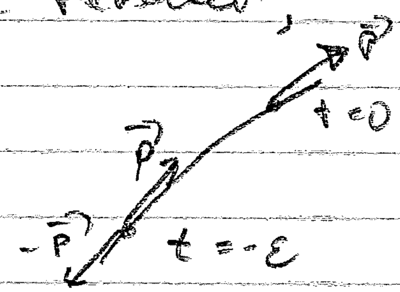
$$|\alpha, \varepsilon\rangle = \left(1 - i\frac{\varepsilon H}{\hbar}\right)|\alpha, 0\rangle$$

time reversed state $\vec{p} \to \vec{p}\,' = -\vec{p}$ evolved from $t=0$ to $t=+\varepsilon$



$$\left(1 - i\frac{\varepsilon H}{\hbar}\right)\hat{T}|\alpha, 0\rangle$$

must equal state at $t=-\varepsilon$ that is time reversed,



$$T\left(1 + \frac{i\varepsilon H}{\hbar}\right)|\alpha, 0\rangle$$

which means

$$-i\varepsilon H T = T i\varepsilon H$$

We cannot have

$$-i\hat{H}\hat{T} = \hat{T}(i\hat{H}) \not\rightarrow -\hat{H}\hat{T} = \hat{T}\hat{H}$$

because $\hat{T}^{-1}\hat{H}\hat{T} = -\hat{H}$ would give for free particles

$$\frac{\hat{p}^2}{2m} \xrightarrow[\hat{T}]{} -\frac{p^2}{2m}$$ which is wrong.

$\hat{T}$ must be anti-unitary, equivalent to

$$\hat{T} = \hat{T}_U \hat{K} \qquad \hat{T}_U^{\dagger} = \hat{T}_U^{-1} \text{ unitary}$$

and complex conjugation operator

$$c\hat{K} = Kc^* \qquad \text{for any number } c$$

or, just use $\hat{T}c = c^*\hat{T}$

then

$$-i\hat{H}\hat{T} = \hat{T}(i\hat{H}) \Rightarrow \hat{H}\hat{T} = \hat{T}\hat{H}$$

$$\hat{T}^{-1}\hat{H}\hat{T} = \hat{H}$$

preserves commutators,

$$[\hat{x}_i, \hat{p}_j]\,|x\rangle = i\hbar\,\delta_{ij}\,|x\rangle$$

anti unitary

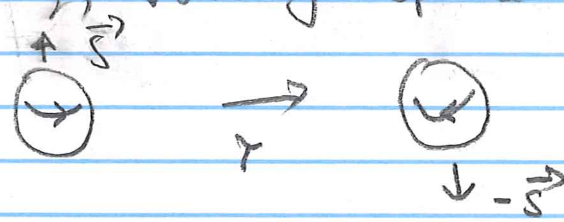
$$\hat{T}\,[\hat{x}_i, \hat{p}_i]\,\hat{T}^{-1}(\hat{T}|x\rangle) = -i\hbar\,\delta_{ij}(\hat{T}|x\rangle)$$

Time reversal of spinor

State $\chi(\hat{n})$ is +1 eigenvalue of $\vec{\sigma}\cdot\hat{n}$ operator. On HW, you show

$$\hat{\tau}\,\chi(\hat{n}) = -i\sigma_2\,\chi^*(\hat{n})$$
$$= \chi(-\hat{n}) \quad \text{flipped spin}$$

classically, rotating sphere



$$(-i\sigma_2)^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -I$$

so $\tau^2\chi(\hat{n}) = -\chi(\hat{n})$

whereas for spin 0 state $\tau^2|0\rangle = +|0\rangle$

generally,

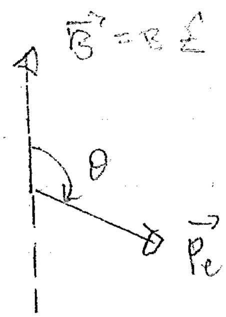
$$\tau^2\,|\tfrac{1}{2}\text{ integer}\rangle = -\,|\tfrac{1}{2}\text{ integer}\rangle$$
$$\tau^2\,|\text{integer}\rangle = +\,|\text{integer}\rangle$$

Parity Violation in β-decay

Wu et al. 1957

Cobalt source at 0.01 K polarized in magnetic field

$$\begin{array}{lccccc} & {}^{60}Co & \rightarrow & {}^{60}Ni^{*} & + \; e^- & \bar{\nu}_e \\ \frac{S_z}{\hbar} & 5 & = & 4 & \frac{1}{2} & \frac{1}{2} \end{array}$$



Observed: $\frac{dN_e}{d\Omega} = 1 - \frac{v}{c}\cos\theta$

Since $\vec{p}_e \xrightarrow[\hat{\Pi}]{} -\vec{p}_e$, this preferred direction violates parity.

Structure of weak interaction.

Weak decay produces left-handed "chiral" electron, spin anti-aligned with momentum.

Electric Dipole Moment of neutron

neutron $(u\,\tfrac{2}{3},\ d\,-\tfrac{1}{3},\ d\,-\tfrac{1}{3})$ quarks

$\tfrac{2}{3}\ \oplus$

$-\tfrac{2}{3}\ \ominus\ \updownarrow \ell = 0.1\,r_n = 0.1 \times fm$

$d_n = q\ell = 4 \times 10^{-4}\ e\text{-cm}$

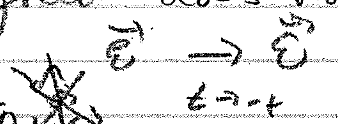
experimentally $d_n < 3 \times 10^{-26}\ e\text{-cm}$

$\vec{d}$ proportional to neutron spin

$$\vec{d} = d\left(\frac{\vec{S}}{\frac{\hbar}{2}}\right)$$

under time reversal, $\vec{S} \to -\vec{S}$ so $\vec{d} \to -\vec{d}$

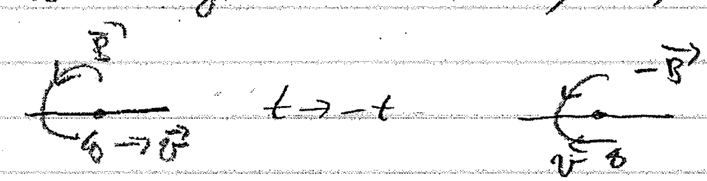
electric field does not change:



$\vec{H}_d = \vec{d}\cdot\vec{\mathcal{E}} \to -\vec{d}\cdot\vec{\mathcal{E}}$ $\cancel{T}$ violation

compare to magnetic moment; $\vec{\mu} = \mu \frac{\vec{S}}{\hbar/2}$

but



So $H_\mu = \vec{\mu}\cdot\vec{B} \to \vec{\mu}\cdot\vec{B}$

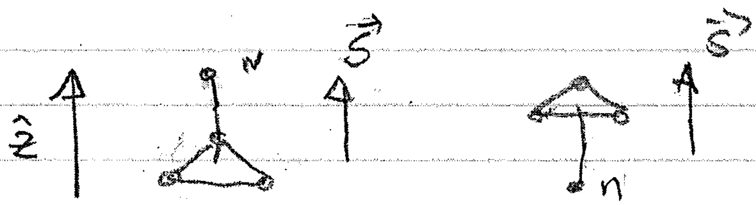
also violates Parity as $\vec{S} \xrightarrow{P} +\vec{S}$ axial-vector

$\vec{d}\cdot\vec{\mathcal{E}} \to -\vec{d}\cdot\vec{\mathcal{E}}$; $\vec{\mu}\cdot\vec{B} \to \vec{\mu}\cdot\vec{B}$ $\quad \vec{\mathcal{E}} \xrightarrow{P} -\vec{\mathcal{E}}$ vector

Note, many molecules have large EDM's, for example

$NH_3 \quad d = 0.3 \times 10^{-8}\ e\cdot cm$

but edm not proportional to spin.
both states exist



$|+z\rangle \qquad\qquad |-z\rangle$

note, ground state has zero electric dipole moment.

<u>note</u> QCD can explain neutron EDM, θ parameter.

θ parameter "naturally" ~ 1

but limit on neutron EDM $\theta < 10^{-10}$

why so small?