Phys 521 Hw #1 Solutions Fall 2023

1.8.5 $\cos\theta \equiv c$, $\sin\theta \equiv s$

(a) $\Omega = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ rotation matrix of x-y plane

$\Omega^{\dagger} = \Omega^{-1} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$$\Omega^{-1}\Omega = \begin{pmatrix} c^2+s^2, & cs-sc \\ cs-sc & s^2+c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) eigenvalue $\Omega|v\rangle = v|v\rangle$

$$\det\left|\Omega - v\hat{I}\right| = 0$$

$$\begin{vmatrix} c-v & s \\ -s & c-v \end{vmatrix} = (c-v)^2 + s^2 = 0$$

$$c - v = \pm i s \Rightarrow v = c \pm is = e^{\pm i\theta}$$

(c) eigenvector

$$\begin{pmatrix} c - e^{\pm i\theta}, & s \\ -s & c - e^{\pm i\theta} \end{pmatrix}\begin{pmatrix} E_+ \\ E_- \end{pmatrix} = 0$$

$$(c - e^{\pm i\theta})E_+ - sE_- = 0$$

$$\frac{E_-}{E_+} = \frac{c - e^{\pm i\theta}}{s} = \frac{\frac{1}{2}(e^{i\theta} + e^{-i\theta}) - e^{\pm i\theta}}{\frac{1}{2i}(e^{i\theta} - e^{-i\theta})} = \pm i$$

then $\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ matrix with eigenvectors as columns

$$\hat{U}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\hat{U}^+ \Omega \hat{U} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} c+is, & c-is \\ -s+ic, & -s-ic \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2(c+is) & 0 \\ 0 & 2(c-is) \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

<u>1.9.1</u> $f(x) = \sum_{n=0}^{\infty} x^n$

Taylor series expansion of $\frac{1}{1-x}$ converges to $f(x)$ for $|x|<1$;

$$f = \frac{1}{1-x} \quad ; \quad f' = \frac{1}{(1-x)^2} \quad ; \quad f'' = \frac{2}{(1-x)^3}$$

$$f^{(n)} = \frac{n!}{(1-x)^{n+1}}$$

$$f(x) = \sum \frac{1}{n!} f^{(n)}(0)\, x^n = \sum x^n$$

remainder of sum $S - S_n = \frac{x^{n+1}}{1-x} \longrightarrow 0$ if $|x|<1$

consider operator defined by series

$$f(\hat{\Omega}) = \sum_{n=0}^{\infty} \hat{\Omega}^n$$

where $\hat{\Omega}$ is Hermitian.
Basis $|\omega\rangle$ with $\hat{\Omega}|\omega\rangle = \omega|\omega\rangle$ ω real

then $$f(\hat{\Omega})|\omega\rangle = \sum_{n=0}^{\infty} \omega^n |\omega\rangle$$

$$= \frac{1}{1-\omega}|\omega\rangle \quad \text{if } |\omega|<1$$

Therefore $f(\hat{\Omega}) = \frac{1}{1-\hat{\Omega}}$ if eigenvalues $|\omega|<1$

1.9.2 Since $[\hat{H}, \hat{H}] = 0$ we can use Baker-Hausdorff Lemma:

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

if $[\hat{A}, [\hat{A},\hat{B}]] = [\hat{B}, [\hat{A},\hat{B}]] = 0$

in this case

$$\hat{U}^+ \hat{U} = e^{-i\hat{H}} e^{i\hat{H}} = e^{i(\hat{H}-\hat{H})} = e^{i0} = \hat{I}$$

Commins 2.2

$u_n(x) = x^n \qquad -1 \le x \le 1$

$$\langle f | g \rangle = \frac{1}{2} \int_{-1}^{1} f(x)\, g(x)\, dx$$

$$\frac{1}{2} \int_{-1}^{1} x^m x^n\, dx = \frac{1}{2} \left(\frac{1}{m+n+1} \right) x^{m+n+1} \Big|_{-1}^{1}$$

$$= \begin{cases} 0 & m+n \text{ odd} \\ \frac{1}{m+n+1} & m+n \text{ even} \end{cases}$$

$x^0 : \phi_0 = 1$

$x^1 : \quad \langle x | x \rangle = \frac{1}{3} \quad \& \quad \langle x | 1 \rangle = 0$

$\phi_1 = \sqrt{3}\, x \qquad \langle \phi_1 | \phi_1 \rangle = 1$

$\langle \phi_1 | \phi_0 \rangle = 0$

$x^2 : \quad a x^2 + b x + c$

$\langle \phi_2 | \phi_0 \rangle = a \langle x^2 | 1 \rangle + c \langle 1 | 1 \rangle$

$= \frac{a}{3} + c = 0 \qquad c = -\frac{a}{3}$

$\langle \phi_2 | \phi_1 \rangle = 0 \implies b = 0$

$\langle \phi_2 | \phi_2 \rangle = a^2 \langle x^2 | x^2 \rangle + \frac{a^2}{9} \langle 1 | 1 \rangle = 1$

$a = 3/\sqrt{10}$

$\phi_2 = \frac{1}{\sqrt{10}} (3x^2 - 1)$

These are Legendre polynomials

4. $$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon}$$

$$\lim x \to 0 \left(\lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon} \right) = \lim_{\epsilon \to 0} \left(\lim_{x \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon} \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} \to \infty$$

normalized Gaussian $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon} dx = 1$

Gaussian with $2\sigma^2 = \epsilon$

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \left(\frac{\epsilon}{x^2 + \epsilon^2} \right)$$

$$\lim_{x \to 0} \lim_{\epsilon \to 0} \left(\quad \right) = \lim_{\epsilon \to 0} \left(\lim_{x \to 0} \frac{1}{\pi} \frac{1}{x^2 + \epsilon^2} \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \to \infty$$

normalization

$$\int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{1}{\pi} \left(\frac{\epsilon}{x^2 + \epsilon^2} \right) dx = \lim_{\epsilon \to 0} \frac{1}{\pi} \tan^{-1}\left(\frac{x}{\epsilon} \right) \Big|_{-\infty}^{\infty}$$

$$= 1$$

5. integrate $\delta(cx)$ over arbitrary $f(x)$

$$\int_{-\infty}^{\infty} f(x)\,\delta(cx)\,dx \underset{y=|c|x}{=} \int_{-\infty}^{\infty} f\left(\frac{y}{c}\right) \frac{\delta(y)}{|c|}\,dy$$

$$= \frac{f(0)}{|c|} \quad \text{so } \delta(cx) = \frac{1}{|c|}\,\delta(x)$$

Commins 2.13

(a) Since $\vec{\sigma}$, $\bar{I}$ span the space of 2x2 matrices, define $\sigma_0 = I$ then arbitrary matrix

$$\bar{m} = \sum_{\mu=0}^{3} m_\mu \sigma_\mu$$

now $[\sigma_0, \sigma_i] = 0$ and

$$[\sigma_i, \sigma_j] = i \sum_k \varepsilon_{ijk} \sigma_k$$

$$0 = [\bar{m}, \sigma_i] = \sum_\mu m_\mu \{\sigma_\mu, \sigma_i\}$$

$$= \sum_{j=1}^{3} m_j \, [\sigma_j, \sigma_i]$$

$$= \sum_{j,k} m_j i \, \varepsilon_{jik} \sigma_k$$

take $i=1$. $i(m_2 \sigma_3 - m_3 \sigma_2) = 0$

true only if $m_2 = m_3 = 0$

similarly for other i, find $m_1 = m_2 = m_3 = 0$

Thus, other than $\bar{I}$, no 2x2 matrix anti-commute with all 3.

2.13 b

from $[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk} \sigma_k$

we see 1) impossible for all 3 to be real

2) take σ_1, σ_2 pure imaginary

then $[\sigma_1 \sigma_2]$ is real and cannot equal $i\sigma_3$ with σ_3 real

(c)

$$(\sigma \cdot A)(\sigma \cdot B) = \sum_{i,j} (\sigma_i \sigma_j A_i B_j)$$

$$= \frac{1}{2} \sum_{i,j} \left(\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j] \right) A_i B_j$$

$$= \frac{1}{2} \sum_{i,j} \left(\sum_k 2i \varepsilon_{ijk} \sigma_k A_i B_j + 2 A_i B_j \delta_{ij} \right)$$

$$= \vec{A} \cdot \vec{B} + i \vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

2.13 d

$$\exp(i\vec{\sigma}\cdot\hat{n}\,\theta) = \sum_{i=0}^{\infty} \frac{1}{n!}(i\vec{\sigma}\cdot\hat{n}\theta)^n$$

with $\sigma_i^2 = I$

$$= I\sum_{n\,\text{even}} \frac{1}{n!}\theta^n + i\vec{\sigma}\cdot\hat{n}\sum_{\text{odd}}\frac{1}{n!}\theta^n$$

$$= I\cos\theta + i\,\vec{\sigma}\cdot\hat{n}\sin\theta$$