HW #1 Problems
Quantum 521

1. Consider the matrix that rotates a Euclidean vector in the x-y plane,

$$\hat{R}^E(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Find the (complex) eigenvalues and corresponding eigenvectors. Write the simlarity transformation matrix $\hat{S}$ with eigenvectors as columns and show explicitly that $\hat{S}^\dagger \hat{R}^E S$ is diagonal. Show that S is unitary, that is $\hat{S}^\dagger S = \hat{I}$ where $\hat{I}$ is the identity matrix. Find the transformed eigenvectors in the new basis where $\hat{R}^E$ is diagonal.

*Exercise 1.9.1.** We know that the series

$$f(x) = \sum_{n=0}^{\infty} x^n$$

may be equated to the function $f(x) = (1-x)^{-1}$ if $|x| < 1$. By going to the eigenbasis, examine when the q number power series

$$f(\Omega) = \sum_{n=0}^{\infty} \Omega^n$$

of a Hermitian operator Ω may be identified with $(1-\Omega)^{-1}$.

*Exercise 1.9.2.** If H is a Hermitian operator, show that $U = e^{iH}$ is unitary. (Notice the analogy with c numbers: if θ is real, $u = e^{i\theta}$ is a number of unit modulus.)

Figure 1: Shankar 1.9.1,1.9.2

2.2. Consider the functions $u_n(x) = x^n$ with $n = 0, 1, 2, \ldots$. These form a basis for a vector space consisting of all real analytic functions of x on the real line in the interval $-1 \leq x \leq +1$. From the functions $u_n(x) = x^n$, we can form an orthonormal basis of functions $\phi_n(x)$ by means of the Schmidt process. Here we define the scalar product of two "vectors" $f(x), g(x)$ by

$$\langle f | g \rangle = \frac{1}{2}\int_{-1}^{1} f(x) g(x)\, dx$$

Find the orthonormal basis functions ϕ_n, $n = 0, 1, 2, 3$. What well-known functions are these?

Figure 2: Commins 2.2

2.13. The three 2×2 Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

play an important role in many areas of quantum mechanics. The $\sigma_{x,y,z}$ and the 2×2 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

together span the four-dimensional vector space of all 2×2 matrices. It is easy to verify from Pauli spin matrices that

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$

and also that

$$\sigma_x \sigma_y = i\sigma_z$$

and cyclic permutations. These two relations can be combined into the following compact form:

$$\sigma_i \sigma_j = \delta_{ij} I + i\varepsilon_{ijk}\sigma_k$$

where δ_{ij} is a Kronecker delta, and ε_{ijk} is the completely antisymmetric unit 3-tensor. Here $\varepsilon_{ijk} = +1$ for $ijk = 1,2,3$ and even permutations thereof; $\varepsilon_{ijk} = -1$ for odd permutations of 1, 2, 3; and $\varepsilon_{ijk} = 0$ whenever two or more of the indices ijk are equal.

The Pauli spin matrices shown earlier constitute the "standard" representation of the Pauli matrices. Another representation is obtained by writing

$$\sigma'_{x,y,z} = U\sigma_{a,y,z}U^{-1}$$

where U is a unitary 2×2 matrix. Because there are an infinite number of possible unitary 2×2 matrices, there are an infinite number of representations of the Pauli spin matrices.

(a) Show that it is impossible to construct a nonvanishing 2×2 matrix that anticommutes with each of the three Pauli matrices.

(b) Show that it is impossible to find a representation of the Pauli matrices where all three are real or where two are pure imaginary and one is real.

(c) Let us define $\boldsymbol{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k}$, where, as usual, $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x-, y-, and z-axes, respectively, in Euclidean 3-space. Let $\boldsymbol{A}$, $\boldsymbol{B}$ be any two vector operators that commute with $\boldsymbol{\sigma}$. Show that the following identity holds:

$$\boldsymbol{\sigma}\cdot\boldsymbol{A}\,\boldsymbol{\sigma}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{B} + i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{B}$$

(d) Let $\hat{\boldsymbol{n}}$ be a unit vector in an arbitrary direction in Euclidean 3-space and θ an arbitrary angle. Show that

$$\exp(i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\theta) = I\cos\theta + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\theta$$

The preceding two identities are important in many quantum mechanical applications.

Figure 3: Commins 2.13

2. Show that plausible representations of the Dirac delta function are,

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-x^2/\epsilon}$$

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

3. Show that $\delta(cx) = \delta(x)/|c|$ where c is a constant. note that $\delta(-x) = \delta(x)$.