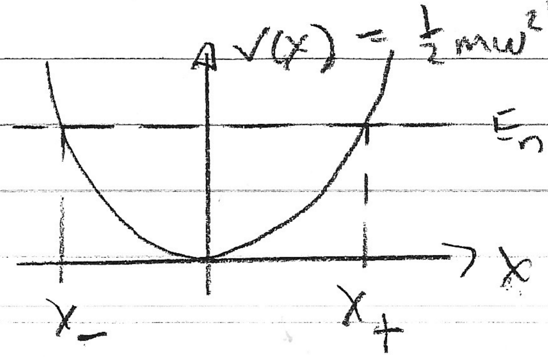
Phys 521 Fall 2025

HW 10a - Solutions

(1) Simple harmonic oscillator by WKB



$$\frac{1}{\hbar}\int_{x_-}^{x_+} \sqrt{2m(E-V(x))}\,dx = \pi\left(n+\tfrac{1}{2}\right)$$

classical turning points $x_\pm = \pm\sqrt{\dfrac{2E}{m\omega^2}}$

$$\frac{1}{\hbar}\int_{-x_+}^{x_+}\left[2m\left(E-\tfrac{1}{2}m\omega^2 x_+^2\left(\tfrac{x}{x_+}\right)^2\right)\right]^{1/2}dx$$

$\frac{1}{2}m\omega^2 x_+^2 = E$ let $y = \dfrac{x}{x_+}$ integral becomes

$$\frac{\sqrt{2mE}}{\hbar}\,x_+\int_{-1}^{1}\sqrt{1-y^2}\,dy = \sqrt{2mE}\,\frac{x_+}{\hbar}\,\frac{\pi}{2} = \left(n+\tfrac{1}{2}\right)\pi$$

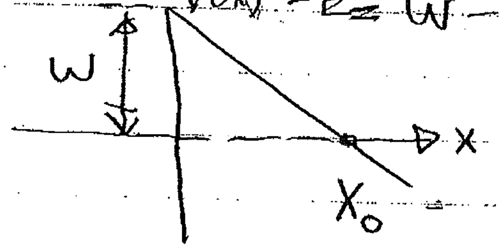
$$\frac{1}{\hbar}\sqrt{2mE}\,x_+\,\frac{\pi}{2} = \frac{1}{\hbar}\left(\frac{2E}{\omega}\right)\frac{\pi}{2} = \left(n+\tfrac{1}{2}\right)\pi$$

$$\boxed{E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega}$$

gives exact energy.

phys522

1) $V(x) - E = W - \alpha x\,, \quad \alpha = e|E|$



the width of the barrier is obtained from

$$W - \alpha X_0 = 0 \quad \Rightarrow \quad X_0 = \frac{W}{\alpha}$$

then the integral for the tunneling factor is:

$$I = \int_0^{X_0} dx \sqrt{\frac{2m}{\hbar^2}(W - \alpha x)}$$

$$= \frac{\hbar^2}{2m\alpha}\int_0^{2mW/\hbar^2} dy\,\sqrt{y} = \frac{\hbar^2}{2m\alpha}\left(\frac{2}{3}\right) y^{3/2}\Big|_0^{2mW/\hbar^2}$$

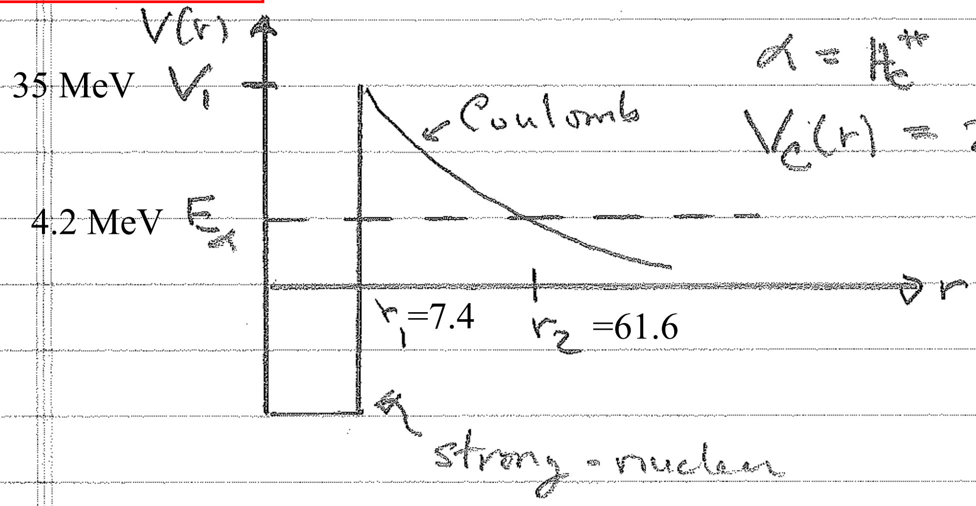
$$= \frac{\hbar^2}{2m\alpha}\left(\frac{2}{3}\right)\left(\frac{2mW}{\hbar^2}\right)^{3/2} = \frac{2\sqrt{2m}}{3\hbar\alpha}W^{3/2}$$

then $T = \exp(-2I) = \exp\left(-\dfrac{4\sqrt{2m}\,W^{3/2}}{3\hbar e|E|}\right)$

exponential dependence on applied voltage

2) alpha decay

$\boxed{\text{values for A=238}}$



$\alpha = He^{++}$ $\quad Z_d = Z-2$

$V_c(r) = 2Z_d e^2/r$

$4 Z_\alpha$

$$E = 2Z_d e^2/r_2 \; ; \; V_1 = 2Z_d e^2/r_1$$

$$r_1 \hat{=} (1.2\,fm)\, A^{1/3} = 7.4 \text{ fm, A=238}$$

for U_{238}, $Z = 92$, $Z_d = 90$. $\boxed{\text{U-238 decays to Th-234 + alpha}}$

$E_\alpha = 4.2$ MeV experimental value

calculate V_1 from r_1

$$V_1 = 2(90)\left(\frac{e^2}{\hbar c}\right)\frac{\hbar c}{r_1} = 2(90)\frac{1}{137}\;\frac{197\,MeV\cdot fm}{7.4\text{ fm}}$$

$$= 35.0 \text{ MeV}$$

calculate r_2 from experimental alpha value

$$V_2 = V(r_2) = E_\alpha = 2Z_d\frac{\hbar c\,\alpha}{r_2}$$

$$r_2 = 2Z_d\,\alpha\,\frac{\hbar c}{E_\alpha} = 2(90)\frac{1}{137}\;\frac{197\,MeV\cdot fm}{4.2\,MeV}$$

$$= 61.6 \text{ fm}$$

Integral over classically forbidden region:

$$g(r) = \frac{\sqrt{2mE}}{\hbar}\sqrt{\frac{V(r)}{E}-1} = \frac{\sqrt{2mE}}{\hbar}\sqrt{\frac{r_2}{r}-1}$$

$$I = \int_{r_1}^{r_2} dr\, g(r) = \frac{\sqrt{2mE}}{\hbar}\left[r_2 \cos^{-1}\sqrt{\frac{r_1}{r_2}} - \sqrt{r_1(r_2-r_1)}\right]$$

for $r_1 \ll r_2$; $\cos^{-1} x \cong \frac{\pi}{2} - x$; $r_1(r_2-r_1) \cong r_1 r_2$

$$I = \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2}r_2 - 2\sqrt{r_1 r_2}\right]$$

with $r_2 = \frac{1}{E}2(Z_d)\hbar c\alpha$ $\quad \propto \sqrt{Z_d}^{4/3}$ because r_1 goes like Zd^(1/3)

$$I = K_1\left[\frac{Z_d}{\sqrt{E}} - \frac{K_2}{K_1}\sqrt{r_1 Z_d}\right]$$

$$K_1 = \alpha\pi\sqrt{2m_\alpha c^2} = 1.980\,(\text{MeV})^{1/2}$$

$$K_2 = 4\sqrt{\frac{\alpha m c^2}{\hbar c}} = 1.486\,(\text{fm})^{-1/2}$$

for large A, $A \approx 2Z_d$

$\frac{K_2}{K_1} = 0.751\,(\text{MeV}\cdot\text{fm})^{-1/2}$; $r_1 = 1.2\,\text{fm}\,A^{1/3} \cong \boxed{1.5\,\text{fm}}\, Z_d^{1/3}$

$$\boxed{I = K_1\left[\frac{Z_d}{\sqrt{E/\text{MeV}}} - Z_d^{2/3}\right]} \cong K_1 X$$

where the factor $\frac{K_2}{K_1}\left(\boxed{1.5\,\text{fm}}\right)^{1/2} = 0.92\,\text{MeV}^{-1/2} \approx 1\,\text{MeV}^{-1/2}$

To get a decay rate, find frequency semi-classically or

$$f = \frac{v}{2r_1} = \frac{c}{2r_1}\sqrt{\frac{2E_\alpha}{m_\alpha c^2}} = \frac{c}{2r_1}\left[\frac{2(4.2\text{ MeV})}{4(931.5\text{ MeV})}\right]^{1/2}$$

$$= 0.0475\ \frac{c}{2(7.4\text{ fm})} = 9.6\times10^{-20}\text{ s}^{-1}$$

for U-238 $Z_d = 92 - 2 = 90$

$$I = k_1\left[\frac{90}{\sqrt{4.5}} - (90)^{2/3}\right] = K_1 \cdot 22.34$$

$$= 44.24$$

$$T = e^{-2I} = 10^{-\log_{10}(e)\,2(44.24)}$$

$$= 10^{-38.4}$$

$$\tau = (fT)^{-1} = 2.6\times10^{17}\text{ s} = 8.3\text{ Gy}$$

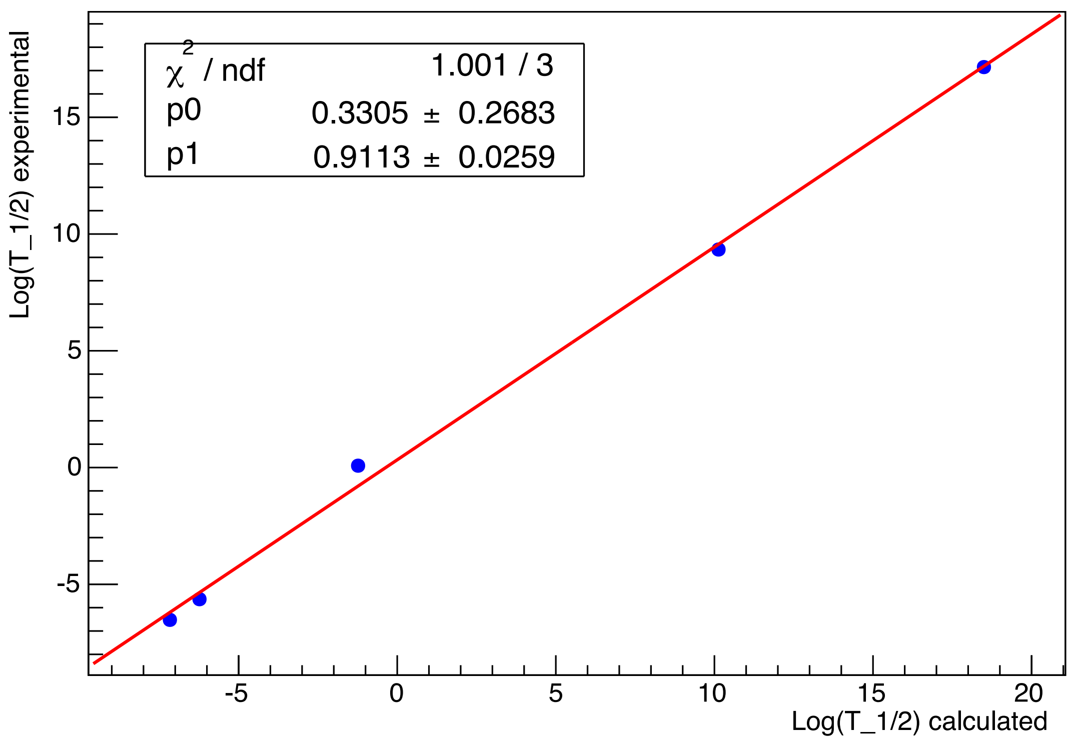
s/y = π 10^7

exp value is 4.468 Gy

Date: September 3, 2015 at 3:08 PM
Topic: alpha decay calculation

nucleus A Z Q half-life computed (experimental)
U-238,A=238, Z=92, Q=4.270 MeV T_1.2 = 3.148E+18 (1.410E+17) seconds
U-232,A=232, Z=92, Q=5.413 MeV T_1.2 = 1.349E+10 (2.173E+09) seconds
Th-215,A=215, Z=86, Q=8.839 MeV T_1.2 = 5.831E-07 (2.300E-06) seconds
Po-212,A=212, Z=84, Q=8.954 MeV T_1.2 = 6.796E-08 (3.000E-07) seconds
Rn-215,A=215, Z=90, Q=7.666 MeV T_1.2 = 5.834E-02 (1.200E+00) seconds

Comparison between this calculation and experimental values:

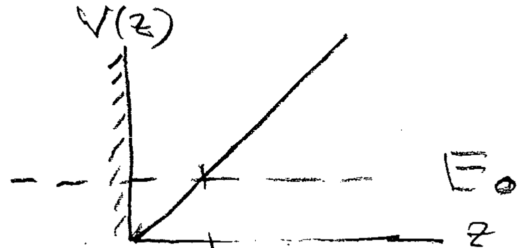


3)

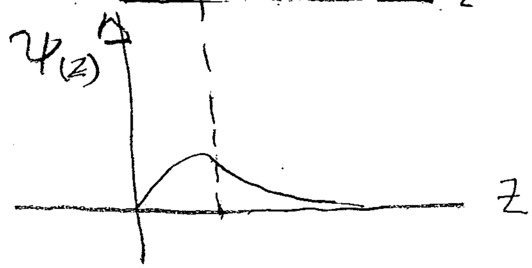
The potential for the particle is

(a) $$V(z) = \begin{cases} mgz & z>0 \\ \infty & z<0 \end{cases}$$

which looks like:



and we expect $\psi(z)$:



$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + mgz \quad \text{w/} \quad \psi(z\leq 0) = 0$$

(b) try $\psi_k(z) = N z e^{-kz}$

the normalization is, $N^2 \int_0^\infty z^2 e^{-2k} \, dz = \frac{N^2}{(2k)^3} 2! = 1$

$$N = 2k^{3/2}$$

$$\frac{d^2}{dz^2}\left(z e^{-kz}\right) = \frac{d}{dz}\left[e^{-kz} - kz e^{-kz}\right] = \frac{d}{dz}\left[(1-kz)e^{-kz}\right]$$

$$= -k e^{-kz} + (1-kz)(-k)e^{-kz}$$

$$= (k^2 z - 2k)e^{-kz}$$

$$\left\langle \frac{d^2}{dz^2} \right\rangle_k = 4k^3 \int_0^\infty dz\, z e^{-kz} (k^2 z - 2k) e^{-kz}$$

$$= 4k^5 \int_0^\infty dz\, z^2 e^{-2kz} - 8k^4 \int_0^\infty dz\, z e^{-2kz}$$

$$= \frac{4k^5}{(2k)^3} 2! - \frac{8k^4}{(2k)^2} = k^2 - 2k^2 = -k^2$$

$$\langle z \rangle_k = 4k^3 \int_0^\infty dz\, z^3 e^{-2kz} = \frac{4k^3}{(2k)^4} 3! = \frac{3}{2} \frac{1}{k}$$

$$\langle \hat{H} \rangle_k = \frac{\hbar^2}{2m} k^2 + mg \left(\frac{3}{2}\right) \frac{1}{k}$$

$$\left. \frac{d \langle \hat{H} \rangle_k}{dk} \right|_{k_m} = \frac{-\hbar^2}{m} k_m - mg \left(\frac{3}{2}\right) \frac{1}{k_m^2} = 0$$

$$k_m^3 = \frac{3}{2} \frac{m^2 g}{\hbar^2}$$

So the estimate of the ground state energy is,

$$\langle \hat{H} \rangle_{k_m} = \frac{\hbar^2}{2m} \left(\frac{3}{2} \frac{m^2 g}{\hbar^2} \right)^{2/3} + mg \left(\frac{3}{2}\right) \left(\frac{\hbar^2}{m^2 g} \frac{2}{3} \right)^{1/3}$$

$$= 3^{2/3} 2^{1/3} \left(\frac{1}{4} + \frac{1}{2} \right) \left(m g^2 \hbar^2 \right)^{1/3} = 1.97 \left(m g^2 \hbar^2 \right)^{1/3}$$

$$\langle z \rangle_{k_m} = \frac{3}{2} \left(\frac{2\hbar^2}{3m^2 g} \right)^{1/3} = \left(\frac{3}{2}\right)^{2/3} \left(\frac{\hbar^2}{m^2 g} \right)^{1/3}$$

What is this for a particle of $m = 1$ gm?

$$mg = (10^{-3}\,\text{kg})(9.8\,\text{m/s}^2) \simeq 10^{-2}\,\text{J/m}$$

$$\frac{\hbar^2}{m} \cong \frac{(10^{-34}\,\text{J.s})^2}{10^{-3}\,\text{kg}} = 10^{-65}\,\text{J}\cdot\text{m}^2$$

$$\langle z \rangle_m \simeq \left(\frac{10^{-65}\,\text{J}\cdot\text{m}^2}{10^{-2}\,\text{J/m}}\right)^{1/3} = 10^{-21}\,\text{m}$$

But for an electron, since $\langle z \rangle \propto m^{-2/3}$

$m_e \approx 10^{-27}\,g$

$$\left(10^{27}\right)^{2/3} = 10^{18}$$

$$\langle z \rangle_m = 10^{-3}\,\text{m} = 1\,\text{mm}.$$

So whereas a macroscopic particle rests on the floor, the electron floats 1mm above the floor.